Orbi Mapping Spaces

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Union College, October 19, 2013

Outline

Orbispaces

Definition

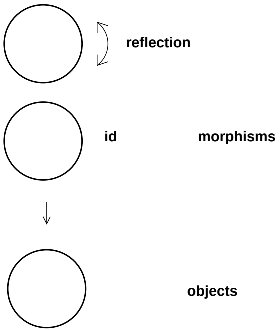
- An **orbispace** is a Morita equivalence class of orbigroupoids.
- An **orbigroupoid** $\mathcal{G}$ is a groupoid in the category of paracompact Hausdorff spaces

$$
\begin{tikzcd}
G_1 \times_{s,G_0,t} G_1 \arrow[r, shift left=2, "\pi_1"] \arrow[r, "m" description] \arrow[r, shift right=2, "\pi_2"'] & G_1 \arrow[r, "i"] & G_1 \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & G_0 \arrow[l, "u" description]
\end{tikzcd}
$$

such that the source and target maps are étale, and the diagonal $(s,t)\colon G_1 \to G_0 \times G_0$ is proper (closed with compact fibers).

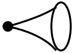
Example 1: The Silvered Interval

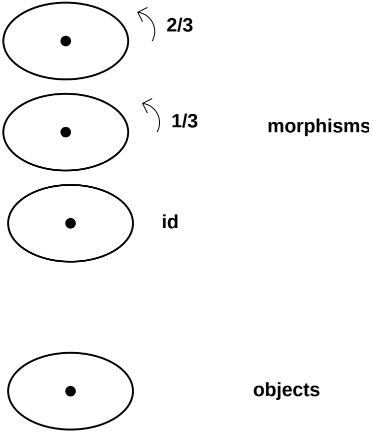
The circle S^1 with the $\mathbb{Z}/2$-action by reflection.



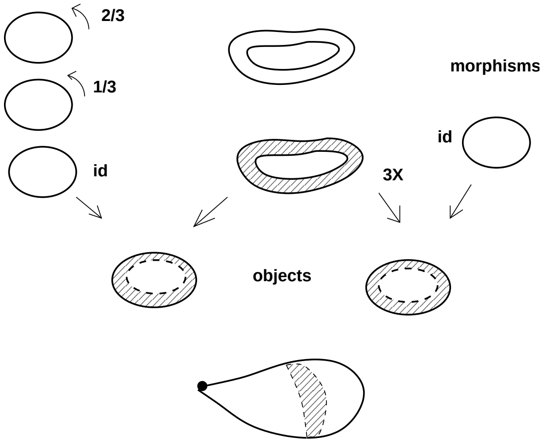
The source map is defined by projection, the target by the action.

Example 2: The Order 3 Cone

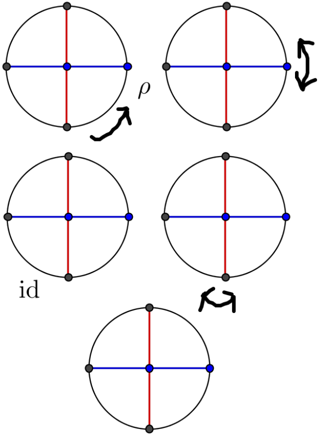




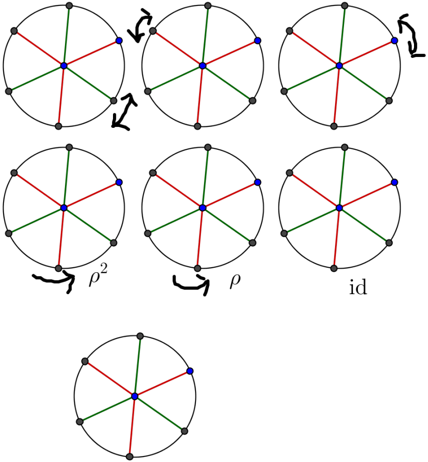
Example 3: The Teardrop



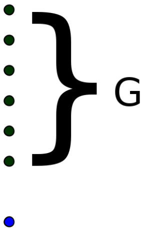
Example 4: The Order 2 Corner $V_4 \ltimes \mathbb{D}$



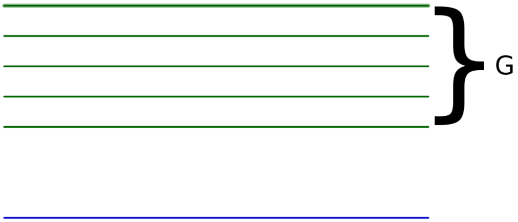
Example 5: The Order 3 Corner $D_6 \ltimes \mathbb{D}$



Example 6: G-Points $*_G$



Example 7: *G*-Lines



Groupoid Maps

Definition

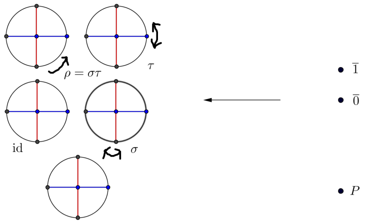
A **morphism** $\varphi\colon \mathcal{G} \to \mathcal{H}$ of topological groupoids is a pair of maps

$$\varphi_0\colon G_0 \to H_0 \text{ and } \varphi_1\colon G_1 \to H_1,$$

which commute with all the structure maps.

$\mathbb{Z}/2$-Points of the Order 2 Corner

- What are the groupoid maps $*_{\mathbb{Z}/2} \to V_4 \ltimes \mathbb{D}$?



- For any $X \in \mathbb{D}$, $\varphi_0^X(P) = X$ and $\varphi_1^X(\bar{0}) = \varphi_1^X(\bar{1}) = (X, \mathrm{id})$.
- For any X on the horizontal (vertical) axis of $\mathbb{D}$, $\psi_0^X(P) = X$ and $\psi_1^X(\bar{1}) = (X, \tau)$ ($\psi_1^X(\bar{1}) = (X, \sigma)$).
- $\chi(P) = O$ and $\chi(\bar{1}) = (O, \rho)$.

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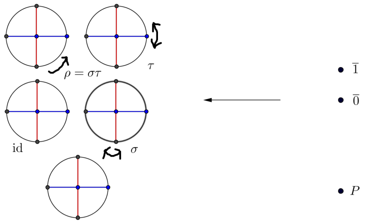

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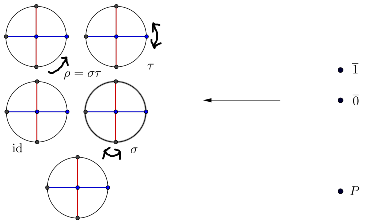

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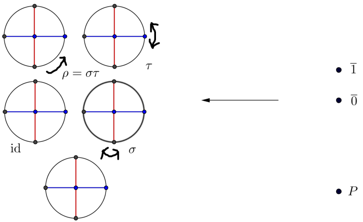

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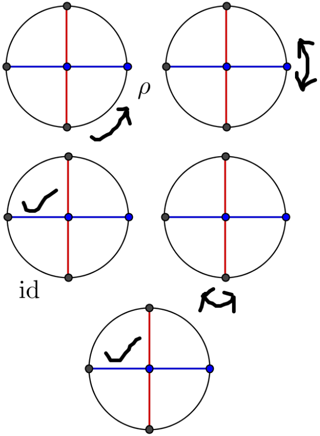
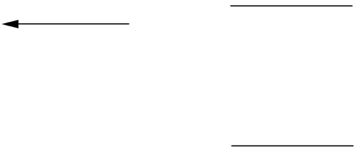
Paths





2-Cells

Definition

A **2-cell** $\alpha\colon \varphi \Rightarrow \psi$ is a map $\alpha\colon G_0 \to H_1$, such that

$$s \circ \alpha = \varphi_0, \quad t \circ \alpha = \psi_0,$$

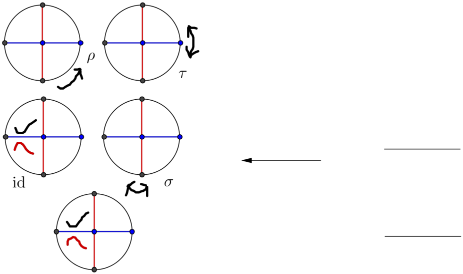
which satisfies the naturality condition, i.e., for each $g \in G_1$,

$$\begin{tikzcd}
\varphi_0(sg) \arrow[r, "\alpha(sg)"] \arrow[d, "\varphi_1(g)"'] & \psi_0(sg) \arrow[d, "\psi_1(g)"] \\
\varphi_0(tg) \arrow[r, "\alpha(tg)"'] & \psi_0(tg)
\end{tikzcd}$$

commutes in $\mathcal{H}$, $m(\psi_1(g), \alpha(sg)) = m(\alpha(tg), \varphi_1(g))$.

2-Cells Between Paths

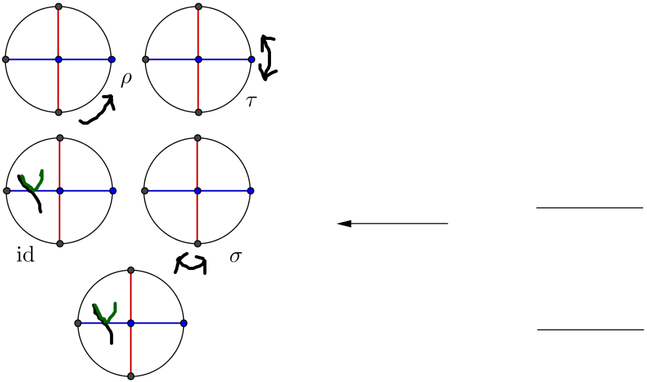
Here are two paths with a unique 2-cell between them.



Note that these paths have the same image in the quotient space.

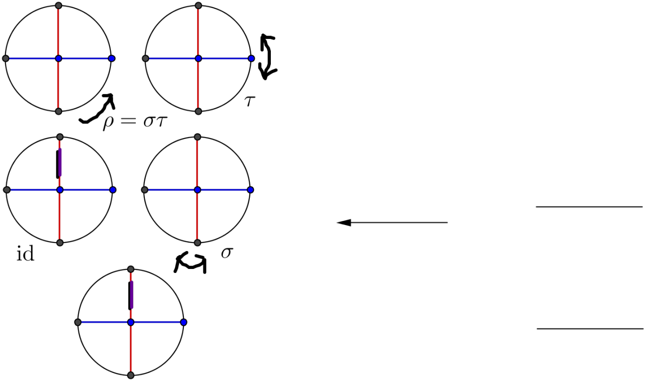
2-Cells Between Paths

These two paths do not have a 2-cell between them, although they have the same image in the quotient space.



2-Cells Between Paths

And these two paths have two 2-cells between them.



2-Cells Between G-Points

If $\varphi(P) = \psi(P)$, then 2-cells

$$\alpha\colon \varphi \Rightarrow \psi\colon *_G \rightrightarrows \mathcal{H}$$

correspond to elements $h \in \mathcal{H}_{\psi(P)}$ such that $h\psi(g)h^{-1} = \varphi(g)$ for all $g \in G$,

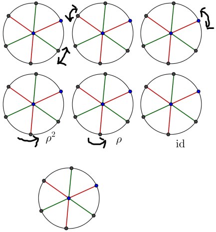
$$\begin{tikzcd} \cdot \arrow[r, "\psi(g)"] \arrow[d, "h"'] & \cdot \arrow[d, "h"] \\ \cdot \arrow[r, "\varphi(g)"'] & \cdot \end{tikzcd}$$

$\mathbb{Z}/3$-Points of the Order 3 Corner

- There are two $\mathbb{Z}/3$ points of the order-3-corner with a non-trivial map on groups:

$$\varphi_0(P) = O,\ \varphi_1(\bar{1}) = \rho \text{ and } \psi_0(P) = O,\ \psi_1(\bar{1}) = \rho^2.$$

- There are three transformations from one to the other (corresponding to the three reflections) and three transformations from each point to itself (corresponding to the rotations).



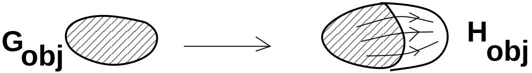
Essential Equivalences, I

An essential equivalence $\phi\colon \mathcal{G} \to \mathcal{H}$ satisfies the following two properties:

1. (**Essentially surjective**)

$$G_0 \times_{H_0} H_1 \longrightarrow H_0$$

is an open surjection,



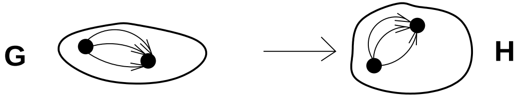
ϕ may not be surjective on objects, but every object in $\mathcal{H}$ is isomorphic to an object in the image of $\mathcal{G}$.

Essential Equivalences, II

2 (**Fully faithful**)

$$\begin{tikzcd} G_1 \arrow[r, "\phi"] \arrow[d, "{(s,t)}"'] & H_1 \arrow[d, "{(s,t)}"] \\ G_0 \times G_0 \arrow[r, "\phi\times\phi"] & H_0 \times H_0 \end{tikzcd}$$

is a pullback,



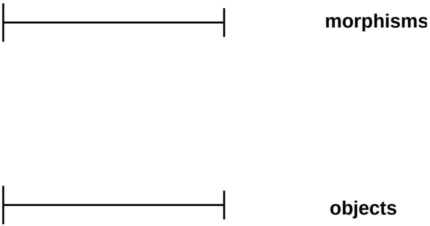
The local isotropy structure is preserved.

Morita Equivalent Groupoids

- Two orbigroupoids $\mathcal{G}$ and $\mathcal{H}$ are called **Morita equivalent** if there exists a third orbigroupoid $\mathcal{K}$ with essential equivalences
$$\mathcal{G} \xleftarrow{\varphi} \mathcal{K} \xrightarrow{\psi} \mathcal{H}.$$
- This is an equivalence relation on groupoids, because essential equivalences of topological groupoids are stable under weak pullbacks (iso-comma-squares).

Examples, I

A line segment can be presented as



Examples, II

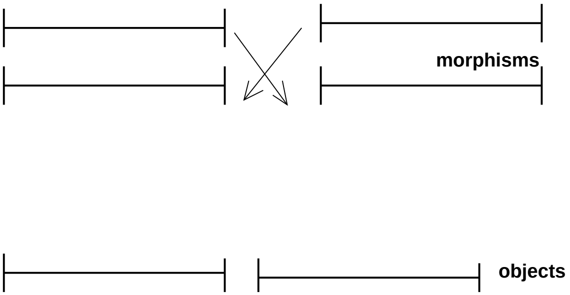
It can also be presented as

morphisms

objects

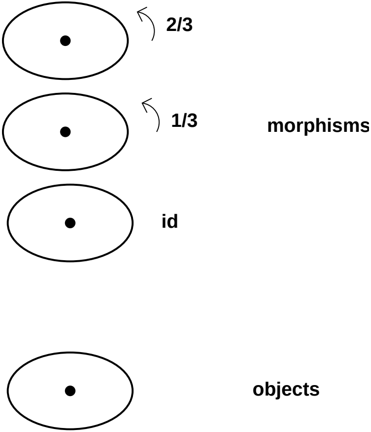
Examples, III

Or as:



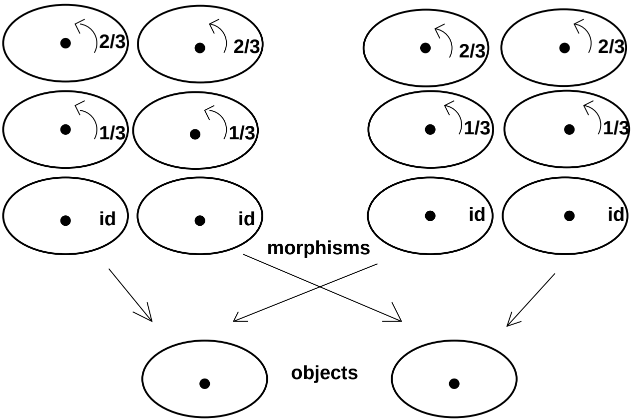
Examples, IV

Here is our order 3 cone again.



Examples, IV

And here is a Morita equivalent presentation



The Bicategory of Orbispaces

Theorem

There is a bicategory of fractions **OrbiGrpd**(W^{-1}) *of orbispaces where:*

- **objects** *are orbigroupoids;*
- **morphisms** *(generalized maps or orbimaps) are spans* $\mathcal{G} \xleftarrow{w} \mathcal{K} \xrightarrow{\phi} \mathcal{H}$ *where w is an essential equivalence;*
- **2-cells** *are equivalence classes of diagrams of the form*

$$\begin{tikzcd} & \mathcal{K} \arrow[dl, "\upsilon"'] \arrow[dr, "\varphi"] & \\ \mathcal{G} \arrow[r, phantom, "\alpha_1\Downarrow"] & \mathcal{L} \arrow[u, "\nu_1"'] \arrow[d, "\nu_2"] \arrow[r, phantom, "\alpha_2\Downarrow"] & \mathcal{H} \\ & \mathcal{K}' \arrow[ul, "\upsilon'"] \arrow[ur, "\varphi'"'] & \end{tikzcd}$$

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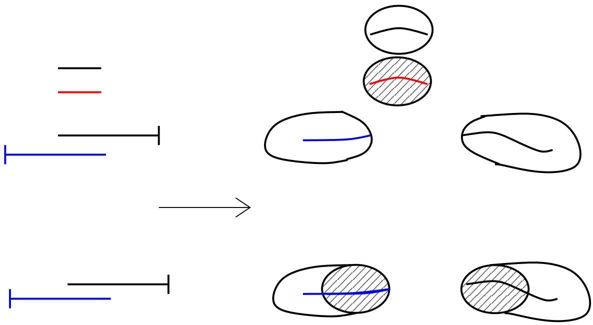
An Example

A map from I to $\mathcal{X}$ (*i.e.*, a path in $\mathcal{X}$):



replacing I by a Morita equivalent orbigroupoid allows us to jump from one chart to another.

Is the category of orbigroupoids with orbimaps Cartesian closed? I.e., can we define an orbi mapping groupoid **OMap**$(\mathcal{G}, \mathcal{H})$?

- Yes, according to Weimin Chen,

  On a notion of maps between orbifolds, I. Function spaces, Comm. Contemp. Math. 8 (2006), no. 5, 569–620

  but the proof is rather messy and intricate.
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  Mapping stacks of topological stacks, arXiv:0809.2373v2

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Orbi Mapping Spaces in Terms of Groupoids

- We want to use the groupoid description of orbispaces to get a description of the orbi mapping spaces as orbigroupoids.
- There are two ways to do this. Both start by first constructing the mapping groupoids for ordinary groupoid homomorphisms.

Let $\mathcal{G}$ and $\mathcal{H}$ be orbi-groupoids. Then the **mapping groupoid**

$$\mathbf{GMap}(\mathcal{G},\mathcal{H})$$

in the category of groupoids and groupoid homomorphisms is described as follows.

- **Space of Objects** $\mathbf{GMap}(\mathcal{G},\mathcal{H})_0$ is the subspace of those f in $\mathbf{Top}(G_1, H_1)$ which preserve composition and units: $m(f,f) = fm$ and

$$\begin{tikzcd} u(G_0) \arrow[d, "f"'] & \subseteq & G_1 \arrow[d, "f"] \\ u(H_0) & \subseteq & H_1 \end{tikzcd}$$

- **Space of Arrows** $\mathbf{GMap}(\mathcal{G},\mathcal{H})_1$ is the subspace of those (f,α) in $\mathbf{GMap}(\mathcal{G},\mathcal{H})_0 \times \mathbf{Top}(G_0, H_1)$ such that

$$sf = s\alpha s \text{ and } tf = s\alpha t.$$

Theorem

- *If $\mathcal{G}$ is a paracompact Hausdorff groupoid such that the space of orbits, G_0/G_1, has finitely many connected components and $\mathcal{H}$ is an orbi-groupoid, then* $\mathbf{GMap}(\mathcal{G},\mathcal{H})$ *is an orbi-groupoid.*
- *For topological groupoids $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{K}$,*

$$\mathbf{TopGpd}(\mathcal{G}\times\mathcal{H},\mathcal{K}) \cong \mathbf{TopGpd}(\mathcal{G},\mathbf{GMap}(\mathcal{H},\mathcal{K})).$$

G-points of various orbi-groupoids

- **GMap**$(*_{\mathbb{Z}/2}, \mathbb{Z}/2 \ltimes S^1)$ is the disjoint union $\mathbb{Z}/2 \ltimes S^1$ with two copies of $*_{\mathbb{Z}/2}$.
- **GMap**$(*_{\mathbb{Z}/2}, V_4 \ltimes D)$ is the disjoint union of $V_4 \ltimes D$, two $\mathbb{Z}_2$-lines which have both an additional $\mathbb{Z}/2$-action by reflection, and a V_4-point.
- **GMap**$(*_{\mathbb{Z}/2}, D_3 \ltimes D)$ is Morita equivalent to the disjoint union of $D_3 \ltimes D$ and a $\mathbb{Z}/2$-line.
- **GMap**$(*_{\mathbb{Z}/3}, D_3 \ltimes D)$ is Morita equivalent to the disjoint union of $D_3 \ltimes D$ and a $\mathbb{Z}/3$-point.

- The $\mathbb{Z}/2$-points of the usual rectangular billiard orbifold (with four order 2 corners) form a disjoint union of the same orbifold, four silvered $\mathbb{Z}/2$-intervals and a V_4-point.
- The $\mathbb{Z}/2$-points of an equilateral triangular billiard orbifold (with three order 3 corners) form the disjoint union of the same orbifold together with a $\mathbb{Z}/2$-circle.
- What would be the $\mathbb{Z}/6$-points of the equilateral triangular billiard orbifold?

The orbi mapping groupoid - option 1

Let $\mathcal{G}$ and $\mathcal{H}$ be orbi-groupoids. To obtain **OMap**$(\mathcal{G}, \mathcal{H})$, the orbi mapping groupoid, we can encode spans $\mathcal{G} \xleftarrow{w} \mathcal{K} \xrightarrow{\phi} \mathcal{H}$ where w is an essential equivalence, for the space of objects, and equivalence classes of diagrams

$$
\begin{tikzcd}
 & \mathcal{K} \arrow[dl, "\upsilon"'] \arrow[dr, "\varphi"] & \\
\mathcal{G} \arrow[r, phantom, "\alpha_1\Downarrow"] & \mathcal{L} \arrow[u, "\nu_1"'] \arrow[d, "\nu_2"] \arrow[r, phantom, "\alpha_2\Downarrow"] & \mathcal{H} \\
 & \mathcal{K}' \arrow[ul, "\upsilon'"] \arrow[ur, "\varphi'"'] &
\end{tikzcd}
$$

for the space of arrows, and show that this gives us again an orbigroupoid.

The orbi mapping groupoid - option 2

Alternatively, we may obtain **OMap**$(\mathcal{G}, \mathcal{H})$ by considering all orbigroupoids **GMap**$(\mathcal{K}, \mathcal{H})$ for essential equivalences $\varphi\colon \mathcal{K} \to \mathcal{G}$, and take a pseudo colimit of these.

The question is: over which diagram?

Essential Equivalences over $\mathcal{G}$

Given an orbigroupoid $\mathcal{G}$, **EssEq**$/\mathcal{G}$ is the 2-category with

- Objects: Essential equivalences $\varphi\colon \mathcal{K} \to \mathcal{G}$.
- Arrows: $(\psi, \alpha)\colon (\mathcal{K}, \varphi) \to (\mathcal{K}', \varphi')$ as in

$$\begin{tikzcd}
\mathcal{K} \arrow[rr, "\psi"] \arrow[dr, "\varphi"'] & \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] & \mathcal{K}' \arrow[dl, "\varphi'"] \\
& \mathcal{G} &
\end{tikzcd}$$

- 2-Cells: $\xi\colon (\psi_1, \alpha_1) \Rightarrow (\psi_2, \alpha_2)$ where $\xi\colon \psi_1 \Rightarrow \psi_2$ is a 2-cell in groupoids, such that

$$\begin{tikzcd}
\mathcal{K} \arrow[rr, bend left, "\psi_2"] \arrow[rr, "\psi_1"'] \arrow[rr, phantom, bend left=20, "\xi\Uparrow"] \arrow[dr, "\varphi"'] & \arrow[d, phantom, "\alpha_1\Uparrow"] & \mathcal{K}' \arrow[dl, "\varphi'"] \\
& \mathcal{G} &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{K} \arrow[rr, bend left, "\psi_2"] \arrow[dr, "\varphi"'] & \arrow[d, phantom, "\alpha_2\Uparrow"] & \mathcal{K}' \arrow[dl, "\varphi'"] \\
& \mathcal{G} &
\end{tikzcd}$$

The Grothendieck Construction

For a small 2-category $\mathbb{D}$ and a 2-functor $J\colon \mathbb{D}^{\mathrm{op}} \to \mathbf{Cat}$, the category $\int_{\mathbb{D}} J$ is defined as follows:

- Objects: (C, x) for $C \in \mathbb{D}_0$ and $x \in J(C)_0$.
- Arrows: equivalence classes of pairs

$$(f, \xi)\colon (C, x) \to (C', x'),$$

where $f\colon C \to C'$ in $\mathbb{D}$ and $\xi\colon x \to Jf(x') = f^*(x')$ in $J(C)$.

- The equivalence relation is generated by: for any 2-cell $a\colon f \Rightarrow g\colon C \rightrightarrows C'$ in $\mathbb{D}$, and any $x \in J(C)$, $x' \in J(C')$,

$$(f, \xi\colon x \to f^*x') \sim (g, (Ja)_{x'} \circ \xi\colon x \to g^*x')$$

Properties of $\int_{\mathbb{D}} J$

- There is an oplax cone $z\colon J\mathbb{D} \to \int_{\mathbb{D}} J$ which gives the oplax colimit of the diagram $J\colon \mathbb{D}^{\mathrm{op}} \to$ **Cat**.
- For $J\colon \mathbb{D}^{\mathrm{op}} \to$ **Grpd**, $\int_{\mathbb{D}} J$ will in general be a category rather than a groupoid.
- When we take the groupoid of fractions of $\int_{\mathbb{D}} J$ for $J\colon \mathbb{D}^{\mathrm{op}} \to$ **Grpd**, we obtain the pseudo colimit of the diagram $J\colon \mathbb{D}^{\mathrm{op}} \to$ **Grpd**.

- In our case $\mathbb{D} = \mathbf{EssEq}/\mathcal{G}$.
- $J\colon \mathbf{EssEq}/\mathcal{G} \to \mathbf{TopGrpd}$ is defined by

$$J\left(\mathcal{K} \xrightarrow{\varphi} \mathcal{G}\right) = \mathbf{GMap}(\mathcal{K}, \mathcal{H})$$

  and J is defined on arrows and 2-cells by composition.
- When we apply the Grothendieck construction with the category of fractions on this diagram we obtain a groupoid which has the property that there is an equivalence of categories from the groupoid encoding the bicategory of fractions diagrams to this new groupoid.
- However, we still need a definition of the topology in the second description.

The Topological Case

- For $J\colon \mathbb{D}^{\mathrm{op}} \to$ **TopGrpd**, we want to do the whole construction inside the world of topological spaces, but we need to use that the diagram $\mathbb{D}$ is a 2-category internal in **Top**.
- **The bad news** In general, we cannot use a Grothendieck construction to construct a fibration out of a family of fibers, and fibrations are not colimits.
- **The good news** We can construct a fibration by Cartesian products if the fibers are constant over the connected components of $\mathbb{D}_0$, and this is the case in our example.
- **More good news** The Cartesian products defined this way do form the space of objects of a topological category that forms the oplax colimit of the original diagram.

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Final results

- The topological category just defined satisfies the conditions to apply the internal category of fractions construction.
- The resulting groupoid is Morita equivalent to the one obtained from the bicategory of fractions; to be precise, the equivalence of categories between the two groupoids mentioned before becomes an essential equivalence of topological groupoids.
- So the hom-categories in the bicategory of fractions may be viewed as homotopy/pseudo colimits, both in the categorical and in the topological case.