## Limit closure of metric spaces

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## What is a uniform space?

Essentially a uniform space is described by the proportion:
topology : uniformity = continuous: uniformly continuous
Created in A. Weil (1938), Sur les espaces à structure uniforme et sur la topologie générale. Act. Sci. Ind. 551, Paris.

## Pseudometrics

A pseudometric $d$ on a set $X$ is a function $d: X \times X \longrightarrow R$ that satisfies:

- $d(x, x)=0$
- $d(x, y)=d(y, x)$
- $d(x, z) \leq d(x, y)+d(y, z)$
- But not $d(x, y)=0$ implies $x=y$


## Uniformities

A (separated) uniformity on a set $X$ is a family $\mathcal{D}$ of pseudometrics on $X$ that satisfies:

- $d \in \mathcal{D}$ and $r>0$ implies $r d \in \mathcal{D}$
- $d, e \in \mathcal{D}$ implies $d \vee e \in \mathcal{D}$
- for $x \neq y \in X$, there is $d \in \mathcal{D}$ such that $d(x, y)>0$

A function $f:(X, \mathcal{D}) \longrightarrow\left(X^{\prime}, \mathcal{D}^{\prime}\right)$ is uniform if for all $d^{\prime} \in \mathcal{D}^{\prime}$, there is a $d \in \mathcal{D}$ such that

$$
d(x, y)<1 \Leftrightarrow d^{\prime}(f x, f y)<1
$$

## Uniform topology

Let $(X, \mathcal{D})$ be a uniform space. For $A \subseteq X, x \in X, d \in \mathcal{D}$, define $d(x, A)=\inf _{a \in A} d(x, a)$. Then $x \in \operatorname{cl}(A)$ if $d(x, A)=0$ for all $d \in \mathcal{D}$. This is a closure operator and defines a topology, called the uniform topology. Distinct uniformities can give the same topology.

## Embedding into Пmetric

For each $d \in \mathcal{D}$ define $E_{d}$ by $x E_{d} y$ if $d(x, y)=0$. Then let $X_{d}=X / E_{d}$ with $q_{d}: X \longrightarrow X_{d}$. Then $d$ induces a metric on $X_{d}$, $q_{d}$ is uniform and $X \hookrightarrow \prod X_{d}$ is an embedding. Thus,

Every (separated) uniform space can be embedded into a product of metric spaces.

## Closed subspaces and equalizers

Suppose $X \subseteq Y$ is closed. Define $f, g: Y \longrightarrow R^{\mathcal{D}}$ by $f(y)(d)=d(y, X)$ and $g(y)(d)=0$. clearly the equalizer of $f$ and $g$ is $X$.
Conversely, since separated uniform spaces are Hausdorff, the equalizer of any two maps $Y \Rightarrow Z$ is closed.

If $X \hookrightarrow Y$ is an embedding of uniform spaces, $X$ is closed in $Y$ if and only if there is an equalizer diagram $X \longrightarrow Y \Longrightarrow Z$

## When is a uniform space closed in a product of metric?

If $(X, \mathcal{D})$ is Cauchy complete (defined next slide), then it is closed in every embedding. But completeness is too strong since every metric space is a closed subspace of a product of metric spaces, namely itself. James Cooper conjectured and we proved that this holds iff every strongly Cauchy net converges.

## Cauchy and strongly Cauchy nets

A net $\left\{x_{i}\right\}$ in $X$ is Cauchy if for all $d \in \mathcal{D}$, there is an $i$ such that $j \geq i$ implies $d\left(x_{i}, x_{j}\right)<1$. The net converges to $x$ if for all $d \in \mathcal{D}$, there is an $i$ such that $j \geq i$ implies $d\left(x_{j}, x\right)<1$. $X$ is complete if every Cauchy net converges.

A net $\left\{x_{i}\right\}$ is strongly Cauchy if for all $d \in \mathcal{D}$ there is an $i$ such that $j>i$ implies $d\left(x_{i}, x_{j}\right)=0$. $X$ is Cooper complete if every strongly Cauchy net converges.

## Limits of metric spaces are Cooper complete

Metric spaces are Cooper complete. One readily sees that the Cooper complete spaces are closed under products and closed subspaces, in particular limits. This shows one half of

A space is a limit of metric spaces if and only if it is Cooper complete.

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## A strongly Cauchy net

Let $y \in \operatorname{cl}(X)$. Define a net $\left\{x_{d}\right\}$ of elements of $X$ indexed by $\mathcal{D}$, which is directed by $\leq$ : choose $x_{d} \in X$ so that $p_{d} y=q_{d} x_{d}$ which is always possible since $q_{d}$ is surjective. Then

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- $\quad=q_{d e} p_{e} y$, since $y \in \operatorname{cl}(X)$


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- and therefore $d\left(x_{d}, x_{e}\right)=0$.


## Conclusion

Thus this is a strongly Cauchy net and therefore converges to some $x \in X$. But it is immediate that $d(x, y)=0$ for all $d \in \mathcal{D}$ and therefore $y=x \in X$ and therefore $X$ is closed in $\prod X_{d}$.

