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The cohomology of symmetric groups and the Quillen map at odd primes

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Abstract

Gunawardena, Lannes, and Zarati proved that the Quillen homomorphism $q_G : H^*BG \to \varprojlim_{\mathcal{C}(G)} H^*BE$ is an isomorphism for $G = \Sigma_n$ at $p = 2$, but fails to be an isomorphism for odd primes. We prove that at odd primes, the restriction of the Quillen map to the subring of elements that are annihilated by all Steenrod operations that involve the Bockstein is an isomorphism for all n.

Key words: group cohomology, symmetric groups, Quillen map, Steenrod algebra, Nil-closed

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1 Introduction

Let p be a prime, let all cohomology be taken with mod p coefficients, and let G be a compact Lie group. Let $\mathcal{C}(G)$ denote the category whose objects

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are elementary abelian p-subgroups of G and whose morphisms are inclusions induced by conjugations in G. Quillen proved that the natural map

$$q_G : H^*BG \to \varprojlim_{\mathcal{C}(G)} H^*BE$$

is an F-isomorphism; that is, the kernel and cokernel are nilpotent as algebras [Q1, Theorem 7.1]. Quillen also established a relationship between this map and the corresponding one for the wreath product $\Sigma_n \int G$: if q_G is a monomorphism, then so is $q_{(\Sigma_n \int G)}$, and thus in particular, q_{Σ_n} is a monomorphism [Q2, Proposition 3.4]. The question of whether q_{Σ_n} is an epimorphism was later settled by Gunawardena, Lannes, and Zarati [GLZ]. For $p = 2$, they proved that if q_G is an isomorphism, then so is $q_{(\Sigma_n \int G)}$, and so q_{Σ_n} is an isomorphism [GLZ, Theorem 1.1]. They further proved that the result fails dramatically at odd primes, even when one makes the standard adjustment of replacing H^*BG by the subring of evenly graded elements [GLZ, Section 6].

For the remainder of the paper, let p be an odd prime. Our goal is to implement a suggestion of David Benson to rectify the odd primary failure of q_{Σ_n} to be an epimorphism (see [B2, p. 175]). Let A denote the mod p Steenrod algebra, and given an unstable A-module M, let M^+ denote the elements of M in even degrees. Let $\tilde{\mathcal{O}}(M)$ be defined by

$$\tilde{\mathcal{O}}(M) = \bigcap_{\theta \in A} \ker(\beta\theta : M^+ \to M);$$

$\tilde{\mathcal{O}}M$ consists of all elements of M that are annihilated by any element of A involving Bocksteins (we call such elements Bockstein-nil) that are also evenly graded. We write $H^\bullet X \equiv \tilde{\mathcal{O}}(H^*X)$ and let

$$q^\bullet_G : H^\bullet BG \to \varprojlim_{\mathcal{C}(G)} H^\bullet BE$$

be induced by the Quillen map.

We will prove the following.

Theorem 1.1 *Let p be an odd prime, and let G be a compact Lie group for which q_G is a monomorphism. If $q^\bullet_G$ is an isomorphism, then so is $q^\bullet_{(\Sigma_n \int G)}$.*

Corollary 1.2 *$q^\bullet_{\Sigma_n}$ is an isomorphism.*

An easy calculation shows that H^*BE has no Bockstein-nil elements in odd degrees, and so whenever Theorem 1.1 applies, the injectivity of q_G implies that $q^\bullet_G$ provides a calculation of the entire Bockstein-nil part of $H^*B(\Sigma_n \int G)$ in terms of Bockstein-nil part of the cohomologies of the elementary abelian p-subgroups of $\Sigma_n \int G$.

From the proof of Theorem 1.1 we are able to extract more information about the full cohomology ring $H^*B\Sigma_n$. Let $\mathcal{N}$ be the two sided ideal of $H^*B\Sigma_n$ consisting of elements x such that $x^n = 0$ for some n.

Proposition 1.3 *Suppose that p is an odd prime, G is a compact Lie group, q_G is a monomorphism, and $q^\bullet_G$ is an isomorphism. If $x \in H^*BG$ is of even degree, then x can be written uniquely as $x = y + z$ where $y \in H^\bullet BG$ and $z^p = 0$.*

Corollary 1.4 *There is an isomorphism of algebras $H^\bullet B\Sigma_n \cong H^*B\Sigma_n/\mathcal{N}$.*

Proof of Corollary 1.4 The odd dimensional part of $H^*B\Sigma_n$ is exterior, and by Corollary 1.2 and Proposition 1.3, the even dimensional part is a direct sum of $H^\bullet B\Sigma_n$ and classes of multiplicative height at most p. However, since q_{Σ_n} is a monomorphism, 2.8(1) implies that the classes in $H^\bullet B\Sigma_n$ have infinite multiplicative height. $\square$

We note that the ideal version of Theorem 1.1 would say that if $q^\bullet_G$ is an isomorphism, then $q^\bullet_{(\Sigma_n \int G)}$ is an isomorphism. In actual fact, Theorem 1.1 has an extra hypothesis, namely that q_G is a monomorphism. This is not needed explicitly by Gunawardena, Lannes, and Zarati, because they assume that q_G is an isomorphism. In our case, the difficulty is that the ideal hypothesis would involve information about $H^\bullet BG$, a submodule of H^*BG, whereas the conclusion would involve information about $H^\bullet B(\Sigma_n \int G)$, which is isomorphic to $H^\bullet(\Sigma_n; (H^*BG)^{\otimes n})$ by a theorem of Nakaoka [N]. Thus the conclusion would involve the full cohomology H^*BG instead of just the submodule $H^\bullet BG$. Assuming that q_G is a monomorphism gives enough extra strength to the hypotheses to allow us to prove Theorem 1.1.

The proof of [GLZ] at $p = 2$ is constructed by "linearizing" the problem. The proof considers the problem in the category $\mathcal{U}$ of unstable A-modules and studies the homological condition of being Nil-closed. The authors prove that q_G is an isomorphism if and only if H^*BG is Nil-closed, and then prove that if H^*BG is Nil-closed, then so is $H^*B(\Sigma_n \int G)$. Our plan of attack is similar. Instead of studying the property of being Nil-closed in the category $\mathcal{U}$, we consider the property of being Nil-closed in the category $\mathcal{U}'$ of evenly graded unstable A-modules.

The organization of the paper is as follows. In Section 2, we give definitions and properties of certain functors of unstable A-modules, and we discuss the concept of a Nil-closed module. In Section 3, we prove Theorem 1.1 and consider its consequences, including a proof of Proposition 1.3. The proofs of two technical results required in the proof of Theorem 1.1 are deferred to Sections 4 and 5. In Section 6, we give a calculation of $H^\bullet\Sigma_{p^2}$.

2 Background and Reformulation

In the first part of this section, we recall straightforward definitions and lemmas regarding unstable modules over the Steenrod algebra and functors applied to them. A useful reference for this material is [S]. In the second part of the section, we discuss the concept of an A-module that is Nil-closed in $\mathcal{U}'$, in parallel to the discussion in Sections 3 and 4 of [GLZ] of modules that are Nil-closed in $\mathcal{U}$.

2.1 The categories $\mathcal{U}$ and $\mathcal{U}'$

Definition 2.1

(1) *$\mathcal{U}$ is the category of unstable modules over the Steenrod algebra.*
$\mathcal{U}' \subseteq \mathcal{U}$ is the full subcategory consisting of evenly graded modules.
$\mathcal{O} : \mathcal{U}' \to \mathcal{U}$ is the forgetful functor.
$\Sigma : \mathcal{U} \to \mathcal{U}$ is the suspension functor.

(2) *If M is an A-module and $x \in M$, we define*

$$P_0x = \begin{cases} P^{|x|/2}x & \text{if } |x| \text{ is even} \\ \beta P^{(|x|-1)/2}x & \text{if } |x| \text{ is odd.} \end{cases}$$

(3) *An unstable A-module N is* **nilpotent** *if for all $x \in N$, $P_0^n x = 0$ for $n >> 0$.*

(4) *A module M in $\mathcal{U}$ (resp. $\mathcal{U}'$) is called* **reduced** *if $\mathrm{Hom}_{\mathcal{U}}(N, M)$ (resp. $\mathrm{Hom}_{\mathcal{U}'}(N, M)$) is zero for all nilpotent modules $N \in \mathcal{U}$ (resp. $\mathcal{U}'$). Equivalently ([S, Lemma 2.6.4]), M is reduced if and only if for all $x \in M$ there exists an operation $\theta \in A$ such that $P_0^n\theta x \neq 0$ for all n.*

(5) *Let M be an unstable A-module. An element $x \in M$ is called* **Bockstein-nil** *if $\beta\theta x = 0$ for all $\theta \in A$. The module M is said to be Bockstein-nil if all its elements are Bockstein-nil.*

Lemma 2.2 *The forgetful functor $\mathcal{O} : \mathcal{U}' \to U$ has a right adjoint $\tilde{\mathcal{O}} : \mathcal{U} \to \mathcal{U}'$, which sends a module to its maximal evenly graded Bockstein-nil submodule.*

PROOF. Routine. □

We will sometimes abuse notation by writing $\tilde{\mathcal{O}}M$ when we should be writing $\mathcal{O}\tilde{\mathcal{O}}M$ and relying on context to make clear which category we are in. The next lemma collects properties of the functor $\tilde{\mathcal{O}}$. The first three properties follow from the definition of $\tilde{\mathcal{O}}$ as a right adjoint.

Lemma 2.3

(1) If M is reduced in $\mathcal{U}$, then $\tilde{\mathcal{O}}M$ is reduced in $\mathcal{U}'$.
(2) If $0 \to A \to B \to C$ is exact, then so is $0 \to \tilde{\mathcal{O}}A \to \tilde{\mathcal{O}}B \to \tilde{\mathcal{O}}C$.
(3) If I is a $\mathcal{U}$-injective, then $\tilde{\mathcal{O}}I$ is a $\mathcal{U}'$-injective.
(4) [LZ86, Proposition 8.3] If M and N are unstable A-modules, and at least one of M and N is reduced, then $\tilde{\mathcal{O}}(M \otimes N) \cong \tilde{\mathcal{O}}M \otimes \tilde{\mathcal{O}}N$.

We will be using the functor $\Phi : \mathcal{U} \to \mathcal{U}$, which functions in some situations as a "pth power" functor and is analogous to the doubling functor mod 2. Given an unstable module M, define

$$(\Phi M)^n = \begin{cases} M^{n/p} & \text{if } n \equiv 0 \ (2p) \\ M^{(n-2)/p+1} & \text{if } n \equiv 2 \ (2p) \\ 0 & \text{otherwise,} \end{cases}$$

and let $\phi x \in \Phi M$ be the element that corresponds to $x \in M$. The action of the Steenrod algebra is given by

$$\begin{aligned} \beta(\phi x) &= 0 \\ P^i(\phi x) &= \begin{cases} \phi(P^{i/p}x) & \text{if } i \equiv 0 \ \ \text{(p)} \\ \phi(\beta P^{(i-1)/p}x) & \text{if } i-1 \equiv 0 \ (p) \text{ and } |x| \equiv 1 \ (2) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

There is a natural map $\lambda_M : \Phi M \to M$ defined by $\lambda(\phi x) = P_0 x$.

Lemma 2.4 *[S, p. 31] If $\lambda_M : \Phi M \to M$ is injective, then M is reduced.*

(The converse is false.)

Both Σ and Φ have right adjoints that will be useful in Section 3 to describe interpretations of Theorem 1.1. We write $\tilde{\Sigma}$ for the right adjoint of Σ, and we note that $\Sigma\tilde{\Sigma}M$ is the largest submodule of M that is a suspension. A module M is reduced if and only if $\tilde{\Sigma}M = 0$. We write $\tilde{\Phi}$ for the right adjoint of Φ, and we think of it as a "pth root functor."

Next, we discuss the homological condition that characterizes whether $q_G^\bullet$ is an isomorphism.

Definition 2.5 *$M \in \mathcal{U}'$ is* **Nil-closed** *in $\mathcal{U}'$ if $\mathrm{Ext}^i_{\mathcal{U}'}(N, M) = 0$ for $i = 0, 1$ for all nilpotent modules $N \in \mathcal{U}'$.*

The importance of a module being Nil-closed comes from the following proposition, which will apply to $q_G^\bullet$ by Proposition 3.1 and Lemma 3.2.

Proposition 2.6 *Let $f : M_1 \to M_2$ be a morphism in $\mathcal{U}'$. Suppose that $\ker(f)$ and $\mathrm{cok}(f)$ are nilpotent modules and that M_2 is Nil-closed in $\mathcal{U}'$. Then the following conditions are equivalent:*

(1) f is an isomorphism;
(2) M_1 is Nil-closed in $\mathcal{U}'$.

PROOF. If M_1 is Nil-closed, then it is reduced, and so $\ker(f)$ is also reduced. Since $\ker(f)$ is assumed to be nilpotent, this says that $\ker(f) = 0$. Thus we have a short exact sequence $0 \to M_1 \to M_2 \to \mathrm{cok}(f) \to 0$, where M_1 and M_2 are both Nil-closed. The long exact sequence in Ext shows that $\mathrm{cok}(f)$ is reduced, and since it is assumed to be nilpotent, this says $\mathrm{cok}(f) = 0$. Thus f is an isomorphism. □

To apply this proposition, it is important for us to be able to tell when a module is Nil-closed, so we next discuss various criteria (see [GLZ]).

Proposition 2.7

(1) [GLZ, Proposition 3.1]
 (a) Let $0 \to M \to M' \to M''$ be an exact sequence in $\mathcal{U}'$. If M' is Nil-closed in $\mathcal{U}'$ and M'' is reduced, then M is Nil-closed in $\mathcal{U}'$.
 (b) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence in $\mathcal{U}'$. If M' and M'' are Nil-closed in $\mathcal{U}'$, then M is Nil-closed in $\mathcal{U}'$.
(2) M is Nil-closed in $\mathcal{U}'$ if and only if M has an injective resolution that begins $0 \to M \to I'_0 \to I'_1$, where I'_0 and I'_1 are reduced injectives in $\mathcal{U}'$.
(3) If M is Nil-closed in $\mathcal{U}$, then $\tilde{O}M$ is Nil-closed in $\mathcal{U}'$.
(4) If M and N are Nil-closed in $\mathcal{U}'$, then so is $M \otimes N$.

The proofs involve standard homological algebra and the following two facts:

- Any reduced module in $\mathcal{U}'$ embeds in a reduced $\mathcal{U}'$-injective.

- The tensor product of two reduced $\mathcal{U}'$-injectives is another $\mathcal{U}'$-injective.

Examples 2.8

(1) $H^\bullet BE$ is the polynomial subalgebra of H^*BE generated by the elements of dimension 2, and it is Nil-closed in $\mathcal{U}'$. Proof: $H^\bullet BZ/p$ is polynomial on a generator of degree 2. Induction on the dimension of E using Lemma 2.3 computes $H^\bullet BE$. To see that $H^\bullet BE$ is Nil-closed in $\mathcal{U}'$, note that $H^\bullet BE = \tilde{O}H^*BE$, and H^*BE is Nil-closed (in fact, injective) in $\mathcal{U}$.

(2) A product of modules that are Nil-closed in $\mathcal{U}'$ is again Nil-closed in $\mathcal{U}'$

(3) An inverse limit of Nil-closed modules is Nil-closed. This is because the inverse limit is the kernel of a map from a Nil-closed module to itself, and we can apply Proposition 2.7 (1a). In particular, we obtain a Nil-closed module by taking the invariants of the action of a group on a Nil-closed module.

3 Proof of Theorem 1.1

The strategy of the proof of Theorem 1.1 is to use Proposition 2.6. Our goal is to prove that if q_G is a monomorphism and if $q_G^\bullet$ is an isomorphism, then

$$q^\bullet_{\Sigma_n \int G} : H^\bullet B\left(\Sigma_n \int G\right) \to \lim_{\substack{\longleftarrow \\ \mathcal{C}(\Sigma_n \int G)}} H^\bullet BE$$

is an isomorphism. We will establish that $q^\bullet_{\Sigma_n \int G}$ has nilpotent kernel and cokernel, that its target is Nil-closed, and that $H^\bullet B(\Sigma_n \int G)$ is Nil-closed. Theorem 1.1 then follows. We note that the only places in the paper where we use the assumption that q_G is a monomorphism are in the last two paragraphs of the proof of Proposition 3.8 and the last paragraph of the proof of Proposition 3.9.

Proposition 3.1 *If G is a compact Lie group, then $q_G^\bullet$ has nilpotent kernel and cokernel.*

PROOF. Consider the following commuting diagram:

$$\begin{array}{ccc} H^\bullet BG & \xrightarrow{q_G^\bullet} & \varprojlim_{\mathcal{C}(G)} H^\bullet BE \\ \downarrow & & \downarrow \\ H^*BG & \xrightarrow{q_G} & \varprojlim_{\mathcal{C}(G)} H^*BE, \end{array}$$

where the vertical arrows are monomorphisms. Certainly $\ker(q_G^\bullet)$ is nilpotent because it is a submodule of the nilpotent module $\ker(q_G)$.

To show that $\mathrm{cok}(q_G^\bullet)$ is nilpotent, recall that we already know that $\mathrm{cok}(q_G)$ is nilpotent by Quillen's theorem. Hence given $y \in \varprojlim_{\mathcal{C}(G)} H^\bullet BE$, there exists $x \in H^*BG$ such that $q_G(x) = y^n$ for some n. We claim that x^p is Bockstein-nil whether or not x is Bockstein-nil. This follows from the calculation

$$\beta\theta x^p = \beta\theta P_0 x = \beta P_0 \theta' x = 0,$$

since $\beta P_0 = 0$, and for any operation θ, there exists θ' with $\theta P_0 = P_0\theta'$. Thus $q_G^\bullet(x^p) = y^{np}$, proving that a power of y is in the image of $q_G^\bullet$. □

Lemma 3.2 $\varprojlim_{\mathcal{C}(\Sigma_n \int G)} H^\bullet BE$ *is Nil-closed in* $\mathcal{U}'$.

PROOF. $H^\bullet BE$ is Nil-closed in $\mathcal{U}'$, and an inverse limit of objects that are Nil-closed in $\mathcal{U}'$ is likewise Nil-closed in $\mathcal{U}'$. (See 2.8.) □

Most of the work of proving Theorem 1.1 lies in showing that $H^\bullet B\,(\Sigma_n \int G)$ is Nil-closed in $\mathcal{U}'$. The proof is by induction, and we begin with $n = p$. Letting $M = H^*BG$, we need to study $H^*(\Sigma_p \int G) \cong H^*(\Sigma_p\,; M^{\otimes p})$ [N] and compute its Bockstein-nil elements. We use the next result to pass the computation through $H^*(Z/p \int G) \cong H^*(Z/p\,; M^{\otimes p})$. For a fixed choice of $Z/p \subseteq \Sigma_p$, let W denote the Weyl group $N_{\Sigma_p}(Z/p)/(Z/p) \cong Z/(p-1)$.

Lemma 3.3 $H^*(B\Sigma_p\,; M^{\otimes p}) \cong H^*(BZ/p\,; M^{\otimes p})^W$.

PROOF. This follows from [B1, Corollary 3.6.19]. □

The module $H^*(BZ/p\,; M^{\otimes p})$ is well understood, and in fact is part of the definition of the Steenrod operations on M in its role as the cohomology of the total space in the fibration sequence $BG^p \to EZ/p \times_{Z/p} BG^p \to BZ/p$. We will need the following maps:

- Let $j : BG^p \to EZ/p \times_{Z/p} BG^p$ be the inclusion of the fiber.
- Let $d : BZ/p \times BG \to EZ/p \times_{Z/p} BG^p$ be defined by the diagonal map on BG.
- Let $\overline{d} : BZ/p \times BG \to E\Sigma_p \times_{\Sigma_p} BG^p$ be the composition of d with the map $EZ/p \times_{Z/p} BG^p \to E\Sigma_p \times_{\Sigma_p} BG^p$ induced by the inclusion $Z/p \hookrightarrow \Sigma_p$.
- Let $\tau_{Z/p} : H^*(* \times BG^p) \to H^*(EZ/p \times_{Z/p} BG^p)$ be the transfer for the inclusion of the trivial group into Z/p.

There is a half-exact sequence

$$0 \to \tau_{Z/p}(M^{\otimes p}) \to H^*(BZ/p\,; M^{\otimes p}) \to H^*(BZ/p) \otimes M, \qquad (3.4)$$

where the monomorphism is given by the inclusion of the image of $\tau_{Z/p}$ and the second map is induced by d^*. (See [S-E, Chapter VII], [Q2, Proposition 3.1], or [Mùi, Theorem 3.7].) According to Lemma 3.3, we must take W-invariants to get to $H^*(B\Sigma_p\,; M^{\otimes p})$.

Let R_1M denote the image of $\overline{d}^* : H^*(E\Sigma_p \times_{\Sigma_p} BG^p) \to H^*(BZ/p \times BG)$. This module was extensively studied and an explicit basis for it described in [Z1], and we summarize the relevant information in Section 4.

Lemma 3.5 *There is a short exact sequence for $H^*(\Sigma_p, M^{\otimes p})$:*

$$0 \to [\tau_{Z/p}(M^{\otimes p})]^W \to H^*(\Sigma_p; M^{\otimes p}) \to R_1M \to 0. \qquad (3.5)$$

PROOF. We take invariants in (3.4) under the action of W. Although taking invariants is only left-exact, we have replaced the right end of the short exact sequence by a module which is defined to make the second map an epimorphism. □

For Theorem 1.1, we actually need to compute the $\tilde{\mathcal{O}}H^*(\Sigma_p; M^{\otimes p})$. In general, $\tilde{\mathcal{O}}$ is not exact (though it is left exact), but the following lemma asserts that it is exact when applied to (3.5).

Lemma 3.7 *Let $M = H^*BG$. There is a short exact sequence*

$$0 \to \tilde{\mathcal{O}}[\tau_{Z/p}(M^{\otimes p})]^W \to H^\bullet(\Sigma_p; M^{\otimes p}) \to \tilde{\mathcal{O}}[R_1M] \to 0.$$

The proof of Lemma 3.7 depends on identifying a particular basis for $\tilde{\mathcal{O}}[R_1M]$ and is given in Section 5.

Proposition 3.8 *If q_G is a monomorphism, and $H^\bullet BG$ is Nil-closed in $\mathcal{U}'$, then so is $H^\bullet B(\Sigma_p \int G)$.*

PROOF. Let $M = H^*BG$. We use Proposition 2.7 (1b) with the short exact sequence of Lemma 3.7. We will prove in Proposition 4.3 that $\tilde{\mathcal{O}}[R_1M]$ is Nil-closed in $\mathcal{U}'$. We must show that $\tilde{\mathcal{O}}[\tau_{Z/p}(M^{\otimes p})]^W$ is Nil-closed in $\mathcal{U}'$. We will actually show that $\tilde{\mathcal{O}}[\tau_{Z/p}(M^{\otimes p})]$ is Nil-closed in $\mathcal{U}'$, and it follows that $\tilde{\mathcal{O}}[\tau_{Z/p}(M^{\otimes p})]^W$ is Nil-closed in $\mathcal{U}'$. (See 2.8). We will be using the fact that $\tau_{Z/p} : (H^*BG)^{\otimes p} \to H^0(Z/p\,; (H^*BG)^{\otimes p})$ is the norm map for Z/p.

First, we claim that $\tilde{\mathcal{O}}[\tau_{Z/p}(M^{\otimes p})] \cong \tau_{Z/p}[(\tilde{\mathcal{O}}M)^{\otimes p})]$. Clearly $\tau_{Z/p}[(\tilde{\mathcal{O}}M)^{\otimes p}] \subseteq \tilde{\mathcal{O}}[\tau_{Z/p}(M^{\otimes p})]$. For the reverse inclusion, note that $\tau_{Z/p}(M^{\otimes p}) \subseteq M^{\otimes p}$, and so

$$\tilde{\mathcal{O}}[\tau_{Z/p}(M^{\otimes p})] \subseteq \tilde{\mathcal{O}}(M^{\otimes p}) \cong (\tilde{\mathcal{O}}M)^{\otimes p},$$

with the last congruence coming from Lemma 2.3, since the assumption that q_G is monomorphic ensures that M is reduced. Then note that $\tau_{Z/p}(M^{\otimes p}) \cap (\tilde{\mathcal{O}}M)^{\otimes p} = \tau_{Z/p}(\tilde{\mathcal{O}}M)^{\otimes p}$.

Let $N \equiv \tilde{\mathcal{O}}M$. Examination of the explicit formula for the cohomology of Z/p with coefficients in the module $N^{\otimes p}$ leads to a short exact sequence

$$0 \to \tau_{Z/p}(N^{\otimes p}) \to (N^{\otimes p})^{Z/p} \to PN \to 0,$$

where PN (the "pth powers" of N) has a basis of the form $n^{\otimes p}$. In our case N is Nil-closed by assumption, so the middle term is Nil-closed because it is obtained from N by tensor product and inverse limit. Further, PN is reduced, because by Lemma 4.5, λ_N is a monomorphism, and then direct calculation shows that λ_{PN} is a monomorphism also. Thus by Proposition 2.7 (1), $\tau_{Z/p}(N^{\otimes p})$ is Nil-closed, and the proof of the proposition is complete. □

The last argument in the proof of Theorem 1.1 is provided by the next proposition, which uses Proposition 3.8 as the beginning of an induction.

Proposition 3.9 *If q_G is a monomorphism, and $H^\bullet BG$ is Nil-closed in $\mathcal{U}'$, then so is $H^\bullet B(\Sigma_n \int G)$.*

PROOF. We write S_nX for the Borel construction $E\Sigma_n \times_{\Sigma_n} X^n$, and given a subgroup $D_n \subseteq \Sigma_n$ and a space X, we write D_nX for the Borel construction $ED_n \times_{D_n} X^n$.

We first treat the proposition in the case $n = p^k$. The p-Sylow subgroup of Σ_{p^k} is contained in the k-fold wreath product

$$D_{p^k} \equiv \left(\Sigma_p \int \Sigma_p \int ... \int \Sigma_p\right) \subseteq \Sigma_{p^k}.$$

Since $H^*S_{p^k}BG$ is a summand in $H^*D_{p^k}BG$, applying $\tilde{\mathcal{O}}$ says that $H^\bullet S_{p^k}BG$ is a summand in $H^\bullet D_{p^k}BG$. However, $D_{p^k}BG \simeq S_pS_p...S_p(BG)$, and by induction with [Q2, Proposition 3.4], we know that $q_{D_{p^k} \int G}$ is a monomorphism. (In particular, $H^*D_{p^k}BG$ is reduced, a fact that we will need in the next paragraph.) Iterating Proposition 3.8 proves $H^\bullet D_{p^k}BG$ is Nil-closed in $\mathcal{U}'$ by induction.

Consider $n = a_0 + a_1 p^1 + ... + a_k p^k$ where $0 \leq a_i < p$. The p-Sylow subgroup of Σ_n is contained in

$$D_n \equiv \prod_{i=0}^{i=k} (D_{p^i})^{a_i},$$

and as before it is sufficient to prove that $H^\bullet D_n BG$ is Nil-closed in $\mathcal{U}'$. Since $D_n BG \simeq \prod_{i=0}^{i=k} [D_{p^i} BG]^{a_i}$, we find that

$$H^* D_n BG \cong \bigotimes_{i=0}^{i=k} [H^* D_{p^i} BG]^{\otimes a_i}.$$

Since $H^* D_{p^i} BG$ is reduced, applying $\tilde{\mathcal{O}}$ to both sides commutes with the tensor product by Lemma 2.3. The proposition then follows from Proposition 2.7 (4). $\square$

Since we know from [GLZ, Section 6] that $\varprojlim_{\mathcal{C}(G)} H^* BE$ is not calculating $H^* B\Sigma_n$, it is reasonable to ask what it is actually calculating. We use the functor $\tilde{\Phi} : \mathcal{U} \to \mathcal{U}$ defined in Section 2.

Proposition 3.10 $\varprojlim_{\mathcal{C}(\Sigma_n)} H^* BE \cong \tilde{\Phi} H^* B\Sigma_n$.

PROOF. Let $\Psi : \mathcal{U} \to \mathcal{U}'$ be the composite $\tilde{\mathcal{O}}\Phi$, and let $\tilde{\Psi}$ be its right adjoint. It is easy to show that $\tilde{\Psi}\tilde{\mathcal{O}} = \tilde{\Phi}$. Applying $\mathrm{Hom}_{\mathcal{U}}(-, H^* BE)$ to the short exact sequence of [S, p. 28] shows that $\tilde{\Phi} H^* BE \cong H^* BE$. Lastly, applying $\tilde{\Psi}$ to both sides to the isomorphism $q^\bullet_{\Sigma_n}$ gives the result. $\square$

For an unstable A-module M, let the Nil-closure of M be its localization away from the subcategory Nil of nilpotent modules in $\mathcal{U}$. The Nil-closure of M is the smallest Nil-closed module containing M, and it is given by the map $M \to \mathrm{colim}_k \tilde{\Phi}^k M$ [S, Theorem 6.3.3].

Corollary 3.11 *$\varprojlim_{\mathcal{C}(\Sigma_n)} H^* BE$ is the Nil-closure of $H^* B\Sigma_n$.*

From Proposition 3.10, we also deduce slightly more information about the Quillen map on $H^* B\Sigma_n$. Quillen's original result shows that q_{Σ_n} is an F-isomorphism, *i.e.,* that $\mathrm{cok}(q_{\Sigma_n})$ is a nilpotent module. Our result implies that $\mathrm{cok}(q_{\Sigma_n})$ is actually a suspension. Let $R^1\tilde{\Sigma}$ be the first right derived functor of $\tilde{\Sigma}$.

Corollary 3.12 $\mathrm{cok}(q_{\Sigma_n}) \cong \Sigma \left[(R^1\tilde{\Sigma}) H^* B\Sigma_n \right]$.

PROOF. If M is a reduced unstable A-module, there is a natural short exact sequence

$$0 \to M \to \tilde{\Phi} M \to \Sigma[(R^1\tilde{\Sigma})M] \to 0$$

[S, p. 38]. The map q_{Σ_n} induces a commuting ladder between the short exact sequence for $M = H^*B\Sigma_n$ and the one for $M = \varprojlim_{\mathcal{C}(\Sigma_n)} H^*BE$. The corollary follows from the Snake Lemma and the isomorphism $\tilde{\Phi} H^*BE \cong H^*BE$. $\square$

We close this section with the proof of Proposition 1.3, which gives us an idea of how much of H^*BG is not captured by $H^\bullet BG$.

Proposition 1.3 *Suppose that p is an odd prime, G is a compact Lie group, q_G is a monomorphism, and $q_G^\bullet$ is an isomorphism. If $x \in H^*BG$ is of even degree, then x can be written uniquely as $x = y + z$ where $y \in H^\bullet BG$ and $z^p = 0$.*

PROOF. First we prove the corresponding result in H^*BE. Suppose that $\{x_1, \ldots, x_n\}$ is a basis for H^1BE and $\{y_1, \ldots, y_n\}$ is a basis for H^2BE, with $y_i = \beta x_i$. We write an even-dimensional element x as the sum of monomials of the form $x_1^{i_1} \ldots x_n^{i_n} y_1^{j_1} \ldots y_n^{j_n}$. Let z be the sum of the monomials in x with $\Sigma i_k > 0$, and let $y = x - z$. Then $z^p = 0$, since each individual monomial in z squares to 0, and y is the sum of terms of the form $y_1^{j_1} \ldots y_n^{j_n}$, which is in $H^\bullet BE$.

Next, we establish the result in $\varprojlim_{\mathcal{C}(G)} H^*BE$. Suppose given an element $(x_E) \in \varprojlim_{\mathcal{C}(G)} H^*BE$, where the notation means a choice of elements $x_E \in H^*BE$ that is compatible over $\mathcal{C}(G)$. For each E, write $x_E = y_E + z_E$, where $y_E \in H^\bullet BE$ and $z_E^p = 0$. We claim that the families y_E and z_E are compatible. To show this, consider elementary abelian subgroups E and F, and let $i : E \cap F \to E$ and $j : E \cap F \to F$ be the inclusions. It is sufficient to show that $i^*y_E = j^*y_F$ and $i^*z_E = j^*z_F$. However, we know that $i^*x_E = j^*x_F$, and so $i^*y_E - j^*y_F = j^*z_F - i^*z_E$. Since the left side is an element of the reduced module $H^\bullet B(E \cap F)$ and the right side is nilpotent, $i^*y_E - j^*y_F = j^*z_F - i^*z_E = 0$. Thus (y_E) and (z_E) are elements of $\varprojlim_{\mathcal{C}(G)} H^*BE$, and the same argument shows that (y_E) and (z_E) are unique.

Finally, consider an even-dimensional $x \in H^*BG$ and let $q_G(x) = (x_E) = (y_E) + (z_E)$ as above. Since $q_G^\bullet$ is an isomorphism, we can find a unique $y \in H^\bullet BG$ with $q_G(y) = (y_E)$. Let $z = x - y$ and observe that $q_G(z)^p = (z_E)^p = 0$, which proves that $z^p = 0$, since q_G is a monomorphism. Uniqueness decomposition of x follows from uniqueness of (y_E) and (z_E) and q_G being a monomorphism. $\square$

4 The module R_1M

In this section, we take an unstable A-module M and study the module R_1M, for which we give a definition in this section. If $M = H^*X$, then R_1M has a geometric interpretation. Let $\overline{d} : BZ/p \times X \to E\Sigma_p \times_{\Sigma_p} X^p$ be defined by the inclusion $Z/p \to \Sigma_p$ and the diagonal map $X \to X^p$. Then $R_1M \cong \mathrm{im}[\overline{d}^* : H^*(E\Sigma_p \times_{\Sigma_p} X^p) \to H^*(BZ/p \times X)]$. Its relevance for us is that if $M = H^*BG$, then R_1M gives part of the cohomology of $H^*B(\Sigma_p \int G)$.

The goal for this section is to prove Proposition 4.3. First we summarize from [Z1] the definitions for R_1M for an unstable A-module M. Let $H \equiv H^*BZ/p$, and identify a basis for $H \cong E[u] \otimes F_p[v]$. Let $w = uv^{-1}$. We write P for the polynomial part of H and note that $P = \tilde{\mathcal{O}}H$.

Definition 4.1 *Given an unstable A-module M, let M^+ be the sub-vector space of elements of even degree, and let M^- be the sub-vector space of elements of odd degree.*

Definition 4.2

(1) We define a Z/p-linear map $St_1 : M \to H \otimes M$ by

(a) If $|x| = 2k$, then

$$St_1(x) = \Sigma_{i=0}^{k}(-1)^i v^{i(p-1)} \otimes P^{k-i}x + (-1)^{i+1} w v^{i(p-1)} \otimes \beta P^{k-i}x$$

(b) $|x| = 2k+1$, then

$$St_1(x) = \Sigma_{i=0}^{k}(-1)^i v^{i(p-1)+(p-1)/2} \otimes P^{k-i}x + (-1)^{i+1} w v^{i(p-1)+(p-1)/2} \otimes \beta P^{k-i}x$$

(2) Let $\mathbf{Z}/p$ be the trivial Σ_p-module, and let $\mathcal{Z}/p$ be the Σ_p-module by the sign representation Let $L_1 \subseteq H$ be defined by $L_1 \equiv \mathrm{im}[H^*(B\Sigma_p; \mathbf{Z}/p) \to H^*(BZ/p; \mathbf{Z}/p)]$ and let $\mathcal{L}_1 \equiv \mathrm{im}[H^*(B\Sigma_p; \mathcal{Z}/p) \to H^*(BZ/p; \mathcal{Z}/p)]$. Then

$$L_1 = wv^{p-1}Z/p[v^{p-1}] \oplus Z/p[v^{p-1}]$$
$$\mathcal{L}_1 = wv^{(p-1)/2}Z/p[v^{p-1}] \oplus v^{(p-1)/2}Z/p[v^{p-1}]$$

(3) [Z1, Definition 2.4.5] Let R_1M be the subvector space of $H \otimes M$ given by

$$R_1M \equiv L_1St_1M^+ \oplus \mathcal{L}_1St_1M^-$$

Although it is not evident from the definition, [Z1, Proposition 2.4.6] establishes that R_1M is an A-submodule of $H \otimes M$. Our interest in the functor R_1 is that Proposition 4.3 is an essential ingredient in the proof of Proposition 3.8, and thus, together with Lemma 3.7, it provides the technical underpinning for Theorem 1.1.

Proposition 4.3 *If $\tilde{\mathcal{O}}(M)$ is Nil-closed in $\mathcal{U}'$, then $\tilde{\mathcal{O}}(R_1M)$ is Nil-closed in $\mathcal{U}'$.*

PROOF. Consider the short exact sequence

$$0 \to R_1M \to H \otimes M \to (H \otimes M)/R_1M \to 0.$$

$\tilde{\mathcal{O}}$ is left exact and commutes with tensor product if one factor is reduced ([LZ86, Proposition 8.3]):

$$0 \to \tilde{\mathcal{O}}(R_1M) \to P \otimes \tilde{\mathcal{O}}(M) \to \tilde{\mathcal{O}}\left[(H \otimes M)/R_1M\right].$$

Since $P = \tilde{\mathcal{O}}H$ is Nil-closed in $\mathcal{U}'$ by Proposition 2.7 (3), and since $\tilde{\mathcal{O}}(M)$ is Nil-closed in $\mathcal{U}'$ by assumption, then $P \otimes \tilde{\mathcal{O}}(M)$ is Nil-closed in $\mathcal{U}'$ by Proposition 2.7 (4). Therefore Proposition 4.3 will follow from Proposition 2.7 (1a) once we prove that $C \equiv \mathrm{cok}[\tilde{\mathcal{O}}(R_1M) \to P \otimes \tilde{\mathcal{O}}(M)]$ is reduced. It is sufficient to prove that $\lambda_C : \Phi C \to C$ is a monomorphism ([S, p39]). Considering the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi\tilde{\mathcal{O}}(R_1M) & \longrightarrow & \Phi[P \otimes \tilde{\mathcal{O}}(M)] & \longrightarrow & \Phi C & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \lambda_{\tilde{\mathcal{O}}(R_1M)}} & & \Big\downarrow{\scriptstyle \lambda_{P\otimes\tilde{\mathcal{O}}(M)}} & & \Big\downarrow{\scriptstyle \lambda_C} & & \\ 0 & \longrightarrow & \tilde{\mathcal{O}}(R_1M) & \longrightarrow & P \otimes \tilde{\mathcal{O}}(M) & \longrightarrow & C & \longrightarrow & 0, \end{array} \tag{4.4}$$

we see that the proposition follows from the Snake Lemma and the two conditions

(1) $\lambda_{P\otimes\tilde{\mathcal{O}}(M)}$ is a monomorphism, and
(2) $\mathrm{cok}\,\lambda_{\tilde{\mathcal{O}}(R_1M)} \to \mathrm{cok}\,\lambda_{P\otimes\tilde{\mathcal{O}}(M)}$ is a monomorphism,

which are proved in Corollary 4.6 and Lemma 4.7, respectively. □

Lemma 4.5 *If M is an unstable A-module that is Bockstein-nil and reduced, then λ_M is a monomorphism.*

PROOF. We know that for all nonzero $x \in M$, there exists $\theta \in A$ such that $P_0^n\theta x \neq 0$ for all n, and since $\lambda_M : \Phi M \to M$ takes $\phi x \mapsto P_0x$, we must prove that in fact $P_0x \neq 0$. Because M is Bockstein-nil, the operation θ cannot involve any Bocksteins. However, we recall that $P_0P^ix = P^{ip}P_0x$, and therefore if $P_0^n\theta x \neq 0$ for all n, it must in fact be true that $P_0^nx \neq 0$ for all n, and in particular, $P_0x \neq 0$. The lemma follows. □

Corollary 4.6 *If $\tilde{\mathcal{O}}M$ is reduced, then $\lambda_{P\otimes\tilde{\mathcal{O}}M}$ is a monomorphism.*

Lemma 4.7 *If $\tilde{\mathcal{O}}M$ is reduced, then* $\operatorname{cok}\lambda_{\tilde{\mathcal{O}}(R_1M)} \to \operatorname{cok}\lambda_{P\otimes\tilde{\mathcal{O}}M}$ *is a monomorphism.*

PROOF. Referring back to diagram (4.4), we see that because $\lambda_{P\otimes\tilde{\mathcal{O}}M}$ is a monomorphism by Corollary 4.6, and thus $\lambda_{\tilde{\mathcal{O}}(R_1M)}$ is likewise, the desired conclusion is equivalent to the requirement that

$$\tilde{\mathcal{O}}(R_1M) \cap \Phi(P\otimes\tilde{\mathcal{O}}M) = \Phi\tilde{\mathcal{O}}(R_1M).$$

Certainly $\Phi\tilde{\mathcal{O}}(R_1M) \subseteq \tilde{\mathcal{O}}(R_1M)\cap\Phi[P\otimes\tilde{\mathcal{O}}M]$. To prove $\tilde{\mathcal{O}}(R_1M)\cap\Phi[P\otimes\tilde{\mathcal{O}}M] \subseteq \Phi\tilde{\mathcal{O}}(R_1M)$, we prove the following conditions:

(1) $\tilde{\mathcal{O}}(R_1M)\cap(P\otimes\Phi(\tilde{\mathcal{O}}M)) = \tilde{\mathcal{O}}R_1(\Phi[\tilde{\mathcal{O}}M])$.
(2) $\tilde{\mathcal{O}}R_1(\Phi[\tilde{\mathcal{O}}M])\cap[(\Phi P)\otimes\tilde{\mathcal{O}}M] \subseteq \Phi\tilde{\mathcal{O}}(R_1M)$.
(3) $[(\Phi P)\otimes\tilde{\mathcal{O}}M]\cap(P\otimes\Phi(\tilde{\mathcal{O}}M)) = \Phi(P\otimes\tilde{\mathcal{O}}M)$.

To establish Condition (1), we use [Z1, 3.3.6], which states that for $M'\subseteq M$, $(R_1M)\cap(H\otimes M') = R_1M'$. Since $\lambda_{\tilde{\mathcal{O}}M}$ is a monomorphism by Lemma 4.5, Condition (1) follows by applying $\tilde{\mathcal{O}}$ to this equality with $M' = \Phi\tilde{\mathcal{O}}M$ and using Lemma 2.3. Condition (3) follows from the fact that $\Phi(P\otimes\tilde{\mathcal{O}}M) \cong \Phi P\otimes\Phi\tilde{\mathcal{O}}M$ (see [S, p.70]). Thus we focus on Condition (2).

We begin by computing $R_1(\Phi[\tilde{\mathcal{O}}M])$. Since $\tilde{\mathcal{O}}M$ is evenly graded, typical elements in $\Phi(\tilde{\mathcal{O}}M)$ are ϕx in dimension $2pk$, where $|x| = 2k$. There are no Bocksteins in $\Phi(\tilde{\mathcal{O}}M)$, and so a typical generator of $R_1[\Phi(\tilde{\mathcal{O}}M)]$ as a module over H is

$$St_1(\phi x) = \Sigma_{i=0}^{kp}(-1)^{-i}v^{i(p-1)}\otimes P^{kp-i}(\phi x).$$

In fact, however, ϕx can only support P^{kp-i} if $kp-i\equiv 0(p)$, and so we can rewrite the typical generator in the form

$$St_1(\phi x) = \Sigma_{j=0}^{k}(-1)^{-j}v^{jp(p-1)}\otimes P^{(k-j)p}(\phi x).$$

Thus a typical Z/p-basis element $y\in R_1[\Phi(\tilde{\mathcal{O}}M)]$ has one of the two forms y_1 or y_2 below:

$$\begin{aligned} y_1 &= \ \ v^{m(p-1)}\left[\Sigma_{j=0}^{k}(-1)^{j}v^{jp(p-1)}\otimes P^{(k-j)p}(\phi x)\right] \\ y_2 &= wv^{m(p-1)}\left[\Sigma_{j=0}^{k}(-1)^{j}v^{jp(p-1)}\otimes P^{(k-j)p}(\phi x)\right]. \end{aligned}$$

When we intersect $R_1[\Phi(\tilde{\mathcal{O}}M)]$ with $(\Phi P)\otimes(\tilde{\mathcal{O}}M)$, we discard elements of type y_2, because they are odd-dimensional. If $y_1\in(\Phi P)\otimes(\tilde{\mathcal{O}}M)$ then $m\equiv 0$

mod p, and we can write $y_1 = \phi z_1$ where

$$z_1 = v^{(\frac{m}{p})(p-1)} \left[\Sigma_{j=0}^{k} (-1)^j v^{j(p-1)} \otimes P^{(k-j)} x \right].$$

By assumption, $x \in \tilde{\mathcal{O}}M$ is Bockstein-nil, and therefore $z_1 \in \tilde{\mathcal{O}}R_1M$. Therefore $y_1 \in \Phi\tilde{\mathcal{O}}(R_1M)$, which completes the proof. □

5 Proof of Lemma 3.7

In this section, our goal is the proof of Lemma 3.7. We must prove that the short exact sequence

$$0 \to [\tau_{Z/p}(M^{\otimes p})]^W \to H^*(\Sigma_p, M^{\otimes p}) \to R_1M \to 0$$

remains exact when we apply $\tilde{\mathcal{O}}$. Since $\tilde{\mathcal{O}}$ is left exact, we need only show that $H^*(\Sigma_p, M^{\otimes p}) \to R_1M$ remains an epimorphism after the application of $\tilde{\mathcal{O}}$. Thus we need to identify the Bockstein-nil part of the target, a task performed by Lemma 5.1. Then we prove Lemma 3.7.

Lemma 5.1 *$\tilde{\mathcal{O}}(R_1M) \cong \tilde{\mathcal{O}}R_1(\tilde{\mathcal{O}}M)$.*

PROOF. Let $C(M)$ be the cokernel of the inclusion $\tilde{\mathcal{O}}M \hookrightarrow M$, so that we have a short exact sequence

$$0 \to \tilde{\mathcal{O}}M \to M \to C(M) \to 0. \tag{5.2}$$

We claim that $\tilde{\mathcal{O}}C(M) = 0$. First, note that $M \to C(M)$ is one-to-one in odd dimensions, since $\tilde{\mathcal{O}}M$ is concentrated in even dimensions. Suppose that $x \in \tilde{\mathcal{O}}C(M)$ and that $y \mapsto x$ under $M \to C(M)$. Then for any sequence I, $\beta P^I y \mapsto \beta P^I x = 0$. Since $\beta P^I y$ is odd dimensional, this implies $\beta P^I y = 0$ for all I. Thus $y \in \tilde{\mathcal{O}}M$, and $x = 0$.

Finally, apply the composite of the exact functor R_1 ([Z1, Corollary 3.3.5]) and the half-exact functor $\tilde{\mathcal{O}}$ to (5.2) to obtain

$$0 \to \tilde{\mathcal{O}}R_1(\tilde{\mathcal{O}}M) \to \tilde{\mathcal{O}}(R_1M) \to \tilde{\mathcal{O}}R_1C(M).$$

Because $\tilde{\mathcal{O}}R_1C(M) \subseteq \tilde{\mathcal{O}}(H \otimes C(M)) \cong P \otimes \tilde{\mathcal{O}}C(M) = 0$ by the previous paragraph, we find $\tilde{\mathcal{O}}R_1(\tilde{\mathcal{O}}M) \cong \tilde{\mathcal{O}}(R_1M)$. □

In the preceding lemma, we identified Bockstein-nil elements by applying $\tilde{\mathcal{O}}$ to $H \otimes C(M)$ and using the fact that $\tilde{\mathcal{O}}$ commutes with tensor products if one factor is reduced. One can also detect Bockstein-nil elements using the following criterion, which we include for general interest.

Lemma 5.3 *Let M be an A-module, and suppose that $x \in M$ has the property that $\beta x = 0$ and $\beta P^n x = 0$ for all n. Then x is Bockstein-nil.*

PROOF. The proof that $\beta P^I x = 0$ for any admissible P^I is by induction on the length of I, with the base case provided by the hypothesis. For the inductive step, we note that an admissible monomial $\beta P^{i_1} P^{i_2}$ can be expressed as a sum of terms of the form $P^u \beta P^v$ and βP^w using the Adem relation

$$P^a \beta P^b = \Sigma_{t=0}^{[a/p]} (-1)^{a+t} \binom{(p-1)(b-t)}{a-pt} \beta P^{a+b-t} P^t + \Sigma_{t=0}^{[(a-1)/p]} (-1)^{a+t-1} \binom{(p-1)(b-t)-1}{a-pt-1} P^{a+b-t} \beta P^t.$$

with $a = pi_2$ and $b = i_2 + (i_1 - pi_2)$. □

Proof of Lemma 3.7 To detect the Bockstein-nil elements in $H^*(\Sigma_p, M^{\otimes p})$, we use the monomorphism

$$(j^*, d^*) : H^*(Z/p\,; M^{\otimes p}) \to M^{\otimes p} \oplus (H^* BZ/p \otimes M).$$

(See, for example, [Q2, Proposition 3.1] or [Mùi, Theorem 3.7].)

From (3.4) and (3.5) we have the commuting ladder

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [\tau_{Z/p}(M^{\otimes p})]^W & \longrightarrow & H^*(\Sigma_p, M^{\otimes p}) & \xrightarrow{\overline{d}^*} & R_1 M & \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \tau_{Z/p}(M^{\otimes p}) & \longrightarrow & H^*(Z/p, M^{\otimes p}) & \xrightarrow{d^*} & H^*(BZ/p) \otimes M & \end{array}$$

where the vertical maps are monomorphisms. The functor $\tilde{\mathcal{O}}$ applied to the top row gives a half-exact sequence

$$0 \to \tilde{\mathcal{O}}[\tau_{Z/p}(M^{\otimes p})]^W \to H^\bullet(\Sigma_p, M^{\otimes p}) \to \tilde{\mathcal{O}}[R_1 M].$$

We must show that $\overline{d}^* : H^\bullet(\Sigma_p, M^{\otimes p}) \to \tilde{\mathcal{O}}[R_1 M]$ is onto. By Lemma 5.1, we know $\tilde{\mathcal{O}}[R_1 M] \cong \tilde{\mathcal{O}} R_1[\tilde{\mathcal{O}} M]$, and because $d^* : H^*(\Sigma_p; M^{\otimes p}) \to R_1 M$ is a map of H^*BZ/p-modules, it is sufficient to show that $H^\bullet(\Sigma_p, M^{\otimes p})$ maps onto the elements $St_1(x) \in \tilde{\mathcal{O}}[R_1 M]$ where $x \in \tilde{\mathcal{O}} M$ is Bockstein-nil. It is possible to choose $z \in H^0(\Sigma_p\,; M^{\otimes p})$ with $j^* z = x^{\otimes p} \in M^{\otimes p}$ and $d^* z = \pm \nu(|x|) St_1(x) \in R_1 M$ where $\nu(|x|)$ is a scalar function. (See [S-E], VII.2.3 and the proof of VIII.1.6. The element z is the element called Px in VII.2.

See also [B2] Chapter 4.) Since x is Bockstein-nil, then so are both j^*z and d^*z, by direct calculation and by hypothesis, respectively. Because (j^*, d^*) is a monomorphism, this implies that z is Bockstein-nil, which gives the result. $\square$

6 Calculation of $H^\bullet\Sigma_{p^2}$

In this section, we describe the ingredients one would use to apply Theorem 1.1 to determine the structure of $H^\bullet\Sigma_{p^n}$ for p odd and we perform the calculation for $H^\bullet\Sigma_{p^2}$.

We begin with some preliminaries on the category $\mathcal{C}(\Sigma_{p^n})$. There is a maximal transitive elementary abelian p-group V_n in Σ_{p^n}, which is unique up to conjugacy and is given by letting $(Z/p)^n$ act on its own p^n elements by translation. The Weyl group $W_{\Sigma_{p^n}}(V_n)$ is $GL(n, \mathbf{F}_p)$, and thus $\tilde{\mathcal{O}}\left(H^*V_n\right)^{W_{\Sigma_{p^n}}(V_n)} \cong \left(H^\bullet V_n\right)^{GL(n,\mathbf{F}_p)}$ is the Dickson algebra $D(n)$, which is a polynomial algebra over $\mathbf{F}_p$ with generators $Q_{n,0}, Q_{n,1}, \ldots, Q_{n,n-1}$.

Next we need a list of the conjugacy classes of maximal elementary abelian p-subgroups of Σ_{p^n}. Let $\mathcal{E}(p^n)$ be the set of sequences of positive integers $\gamma = (\gamma_1, \gamma_2, ...)$ such that $p^n = \Sigma\gamma_k p^k$; note that there is no requirement that γ_k be less than p. The set of conjugacy classes of maximal elementary abelian p-subgroups of Σ_{p^n} is in one-to-one correspondance with $\mathcal{E}(p^n)$ by associating to γ the maximal elementary abelian p-subgroup

$$E(\gamma) = \prod_k V_k^{\gamma_k},$$

whose Weyl group is

$$W(\gamma) = \prod_k \Sigma_{\gamma_k} \int GL(k, \mathbf{F}_p).$$

Theorem 1.1 tells us that

$$\begin{aligned} H^\bullet\Sigma_{p^n} &\cong \tilde{\mathcal{O}} \lim_{\substack{\longleftarrow \\ \mathcal{C}(\Sigma_{p^n})}} H^*BE \\ &\cong \lim_{\substack{\longleftarrow \\ \mathcal{C}(\Sigma_{p^n})}} H^\bullet BE, \end{aligned}$$

and thus an element of $H^\bullet\Sigma_{p^n}$ is given by a family of elements, one in each module $(H^\bullet E(\gamma))^{W(\gamma)}$, that are compatible under restriction to elementary abelian p-groups of lower rank. Note that $(H^\bullet E(\gamma))^{W(\gamma)}$ is given by a tensor product of the algebras

$$\left(D(k)^{\otimes\gamma_k}\right)^{\Sigma_{\gamma_k}},$$

which consist of symmetric functions in the Dickson invariants.

To compute $H^\bullet\Sigma_{p^2}$, we are looking at the inverse limit of $H^\bullet$ applied to the diagram

$$\begin{array}{ccc} V_1^p & & V_2 \\ & \nwarrow \quad \nearrow & \\ & V_1 & \end{array}$$

where the maximal nodes have the automorphisms

$$\begin{aligned} \mathrm{Aut}(V_1^p) &\cong \Sigma_p \int GL(1, \mathbf{F}_p) \\ \mathrm{Aut}(V_2) &\cong GL(2, \mathbf{F}_p). \end{aligned}$$

We write $\sigma_i(x_1, ..., x_p)$ for the ith symmetric function, and we write Q_i' for the element of $H^\bullet V_1^p$ which is the tensor product of 1's in each factor except the ith factor, where it is $Q_{1,0}$. Then potential generators of $H^\bullet\Sigma_{p^2}$ are provided by two sets of elements:

(1) the polynomial generators s_i for $i = 1, \ldots p$ where

$$s_i \equiv \sigma_i(Q_1', \ldots, Q_p') \in (H^\bullet V_1^p)^{\mathrm{Aut}(V_1^p)},$$

(2) the polynomial generators $Q_{2,0}$ and $Q_{2,1}$ in $H^\bullet V_2$.

To check compatibility, observe that $V_1 \to V_1^p$ is the diagonal map, and therefore

$$V_1 \to V_1^p \text{ induces} \begin{cases} s_i \mapsto 0 & \text{for } 0 < i < p \\ s_p \mapsto Q_{1,0}^p \end{cases}$$

$$V_1 \to V_2 \text{ induces} \begin{cases} Q_{2,0} \mapsto 0 \\ Q_{2,1} \mapsto Q_{1,0}^p. \end{cases}$$

Checking compatibility, we find that $H^\bullet\Sigma_{p^2}$ is generated by

$$\begin{aligned} x_i &= (s_i, 0) \text{ for } 1 \le i < p \\ v &= (0, Q_{2,0}) \\ w &= (s_p, Q_{2,1}) \end{aligned}$$

Thus $H^\bullet\Sigma_{p^2} \cong \mathbf{F}_p[x_1, ..., x_{p-1}, v, w]$ with the relations $x_i v = 0$.

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