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AN INTERACTIVE GALLERY ON THE INTERNET: "SURFACES BEYOND THE THIRD DIMENSION"

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Originally developed as an exhibition for the Providence Art Clubs' Dodge Gallery in March of 1996, "Surface Beyond the Third Dimension" continues its existence today as a virtual experience on the internet. Here, the authors take you on a tour of the exhibit and describe both the artwork and the mathematics underlying the images, whose subjects range from projections of spheres and tori in four-space, to images of complex functions, to views of the Klein bottle. Along the way, the authors introduce some of the issues that arise from this dual presentation approach, and point out some of the enhancements available in the virtual gallery that are not possible in the physical one.

Keywords: Complex functions, fourth dimension, mathematical art, surfaces, torus, Klein bottle

1. Introduction

What is the best way to display a variety of surfaces so as to encourage many people to interact with them? Stage an exhibit. In March of 1996, the Providence Art Club, one of the oldest such clubs in the country, hosted the show "Surfaces Beyond the Third Dimension" in their Dodge House Gallery. The first incarnation of that exhibition lasted for two weeks, and the gallery book includes the signatures of dozens of visitors, including artists, students, and mathematicians. The physical exhibit has long since been dismantled, yet the show lives on as a virtual experience[a] and we still receive comments in the on-line guest book. In this article, we review the processes used in constructing both the original show and its virtual continuation, and raise a number of questions about the potential of this medium for reaching large numbers of people with many different backgrounds and interests.

[a]See the URL ⟨`http://www.math.brown.edu/~banchoff/art/PAC-9603/`⟩.

2. The Exhibit Then and Now

The Dodge House Gallery has a square base and one interior partition, providing well-lighted wall space for twelve large photographic reproductions of computer graphics images as well as an alcove for display of a continuous videotape featuring two three-minute videos (see Color Figure 1, left). A guidebook gave information about the nature of the objects, including technical descriptions of the software and hardware used in the design of the objects and the Ilfochrome process used in their reproduction. Additional pages gave mathematical descriptions of the various pieces, as well as references to places where they had appeared either in research articles or as illustrations in books and journals. There was also a well-attended afternoon gallery talk, describing the origins of the project and including a guided tour of the exhibit.

All of these aspects of the physical exhibit are enhanced in the virtual counterpart. In a certain sense, the on-line version contains much more than the original. To what extent does it capture and augment the experience of those who visited the actual gallery opening, and came back to see and respond to the images on the walls? There are many questions raised by this means of portraying mathematical art and design, and we will address some of them now.

What is it that we are showing? Most of the images included were originally studied as abstract geometric constructions given by parametric surfaces in three- and four-dimensional space. In some cases, there is an elaborate theory behind the illustration, whereas in other cases, the phenomena are not yet well understood. In several instances, the display itself represents an innovation not only in the method of displaying a surface but in representing its mathematical properties in a way that suggests new results.

3. A Tour of the Exhibit

"Math Horizon" is so named because it appeared as the cover image for an article in the journal "Math Horizons", published by the Mathematical Association of America. The object under investigation is an immersion of the two-dimensional sphere into four-space in such a way that there is exactly one point where the surface intersects itself. In this sense, it represents an analogue of the figure-eight, an immersion of the one-sphere into two-space self-intersecting at a single point. In neither case can the immersion be deformed into an embedding, with no self-intersections, without introducing local singularities such as cusps. The particular image is obtained by projecting centrally from a point on the three-sphere so that the observer appears to be inside the object. Color depicts different circles of latitude on the original two-sphere. The north and south poles are both mapped to the origin in four-space and no other pair of points is mapped to a common point in four-space. Projecting into three-space does introduce a curve of double points, including the image of the origin. The example was originally introduced in a research/expository article on the geometry of characteristic classes for surfaces in four-space written together with Frank Farris [2].

When this image was used as the cover for the journal, it was rotated ninety degrees from how it appeared in the show. The placement on the exhibit wall was voted on by a group of artists at the Art Club. This orientation was preferred since it more closely suggested a sunset over the water. One artist strongly wanted the image to be hung upside down from the position finally chosen, precisely because it presented the same geometrical picture but with an inversion of the expected sunset color values. There is, of course, no right way to hang such an abstract image. One exhibitor at the PAC expressed this ambiguity in his show by mounting some pieces on rotating discs, but this does not work for a basically rectangular piece.

The "Torus Triptych" is an example of commercial art, being new illustrations from the second edition of the author's book *Beyond the Third Dimension* [1]. The various images in each of the three parts (one is shown in Figure 1) indicate the "water-level curves" as a torus is gradually submerged into a liquid, giving quite different collections of curves depending on the way the torus is positioned with respect to a horizontal plane. There are links to that book in the author's bibliography[b] as well as a direct link to Amazon Books, which provides information about it together with an opportunity to order it directly.

Further down on the same wall are three "Tetraviews" (Color Figure 2), each showing an assembly of five images, two smaller squares partially obscured by opposite corners of a large central square, which has its other two corners partially obscured by two squares of medium size. The display is inspired by the work of the artist Hans Hofmann, who began in the Bauhaus School. The five images are different views of a single surface in four-dimensional space, and the four corners show projections into the four coordinate hyperplanes. The dominant fifth image is in equilibrium, in a real sense the average of the other four. The ability to navigate between any two of these views is crucial for the understanding of the surface, according to the article "Understanding Complex Function Graphs" by the authors in the prototype volume of the totally electronic journal *Communications in Visual Mathematics*, sponsored by the Mathematical Association of America and the National Science Foundation. There is a link from each of these pictures to that article.

Related to the Tetraview series is the pair of "Necklace" views on the next wall (Color Figure 1, right, and Figure 6). Once again each subject is the graph of a function of a complex variable, first the complex squaring operation and then the cubing function. In each case there are five images, starting with a disc and ending with a doubly or triply covered disc, with intermediate steps possessing three- or four-fold symmetry. Both pages include links to the "Understanding Complex Function Graphs" article in *Communications in Visual Mathematics* mentioned above. The first of these pictures appeared in rotated form as "Z-Squared Crescent" on the cover of the *Notices of the American Mathematical Society*, with a technical explanation inside, and there is also a link to that publication. A black-and-white

[b] See the URL ⟨`http://www.math.brown.edu/~banchoff/`⟩ for links to the author's annotated bibliography.

version of this same image was used on the postcard invitation to the show, which is reproduced on the opening page of the web site.

Next is one of the most elaborate entries in the exhibit, not because of the complexity of the image but because of the linkages. “Triple-Point Twist” (Figure 2) appeared in slightly rotated form on the cover of the *Notices of the AMS* as well as on the cover of a statistics volume by Gergen and Iversen. As chapter headings in that latter volume there were fourteen views of this object rotating in three-space. These can be accessed in the virtual exhibit either as an MPEG movie or as a virtual reality VRML document, enabling the viewer of the electronic version to interact with the object in ways impossible for the gallery visitor.

Even more significant than watching the object rotate is to view and review an MPEG movie that makes the object unfold, changing one of the parameters that twist the “bamboo curtain” ruled surface so that it intersects itself, forming a triple point. The equations defining the surface have been studied extensively by David Mond and Washington Marar [4], and there are links to several items in their bibliography for the interested mathematician reader.

Once again, it is these enhancements that represent the true innovation in such a virtual gallery. The viewer who becomes fascinated by one or another of the aspects of an object can investigate it at an appropriate level, depending on the background and interests of the individual. In particular, it is possible in some cases to view and manipulate phenomena that relate the particular object to a wider area of mathematics.

The final wall has two parts, the first of which contains a series of three images presented at the celebration of the hundredth birthday of Prof. Dirk Struik. On September 23, 1994, at Brown University, Prof. Struik gave his own centenary lecture “Mathematicians I Have Known”, and the gallery includes a photograph of him during that event. To the left of his picture are three geometric figures developed during the summer of that year by teams of students working at Brown. In the electronic version there is a link to the story about that lecture from the *Notices of the AMS*.

Each of the three images that were presented to Prof. Struik has its own set of links. “The Temple of Viviani” is an enhancement of a standard figure from descriptive geometry and graphical solid modeling, depicting the intersection of a sphere with a circular cylinder of half its radius passing through the center of the sphere and tangent to it at one point of the equator. This example is found in any multivariable calculus book since it is one of the first interesting cases for which it is possible to evaluate both the volume and the surface area of the intersection. It is also one of the most frequently misdrawn illustrations, since many volumes draw only the top half and either make the bottom point of the curve smooth or cuspidal, whereas the computer diagram clearly indicates a figure-eight curve with a transversal crossing. From the point of view of Lagrange multipliers, this position represents the non-transversal intersection of a cylinder with the level set of a distance function to a point (or, dually, the intersection of a sphere with the distance function to a line). This image accentuates the intersection curve by presenting it as a small tube. In the exhibition, there was a mention that the same figure was featured in one of the continuously projected videotapes shown in the alcove of the gallery. The history of the image and its use in calculus was presented in the exhibit booklet, and as a link on the electronic version. The image rendering was done by Julia Steinberger and Neel Madan, while the videotape sequence was designed and executed by Ying Wang.

The second image in this sequence, a ray-traced self-intersecting Klein bottle, also refers to a sequence in the same videotape, a fly-through which intersects the surface at two positions. The equations for this image were developed by David Kaplan and the renderings in the videotape, together with the soundtrack, are the work of Jeff Beall.

The final image of the three presents a study of the evolute surfaces of a right helicoid. This view was created by Cathy Stenson, in connection with her research on the mathematics of DNA coiling.

The last image featured in the exhibit "Surfaces Beyond the Third Dimension" is an interior view of a cyclide of Dupin, a torus on a three-dimensional sphere in four-space projected stereographically from a point on the torus itself, leading to a third-order algebraic surface expressed as a union of circles (and four straight lines). These curves are orbits of a Hamiltonian dynamical system and the fibers over a great circle of the Hopf mapping from the three-sphere to the two-sphere. The exhibit booklet and the electronic links refer the viewer to several articles written by the author and colleagues in Applied Mathematics and Computer Science [3], examining different aspects of this extremely important surface. This image is also featured on the cover of the Scientific American Library volume *Beyond the Third Dimension* [1].

Finally, there is the customary guest book, into which visitors to the actual exhibit wrote their signatures and comments. The electronic version enables visitors to the web site to enter their own comments and to read those of other visitors.

4. The Mathematics Behind the Images

The surfaces presented in the exhibit "Surfaces Beyond the Third Dimension" all represent views of four-dimensional objects, either as projections into three-space of surfaces defined in four-space, or as a sequence of related surfaces in three-space, where time plays the role of the fourth dimension.

The "Torus Triptych" and "Triple-Point Twist" are examples of the latter, and we begin with these. A torus can be generated by rotating a circle around an axis in the same plane as the circle, but not intersecting it. We can produce parametric equations for such a torus of revolution as follows: if we consider a circle of radius b in the xz-plane, centered at the point $(x, z) = (a, 0)$ on the x-axis, then the points on the circle are given by $(x, z) = (a + b\cos\theta, b\sin\theta)$. If we rotate this circle about the z-axis, then each point (x, z) on the original circle traces out a new circle in a plane parallel to the xy-plane; the radius of this new circle will be x (the distance of the original point from the z-axis), and the height of the plane containing the new circle will be z. This means the new circle can be parameterized by $(x \cos\phi, x \sin\phi, z)$. As we let (x, y) vary over the entire original circle, we obtain a parameterization for the torus:

$$T(\theta, \phi) = ((a + b\cos\theta)\cos\phi, (a + b\cos\theta)\sin\phi, b\sin\theta). \tag{1}$$

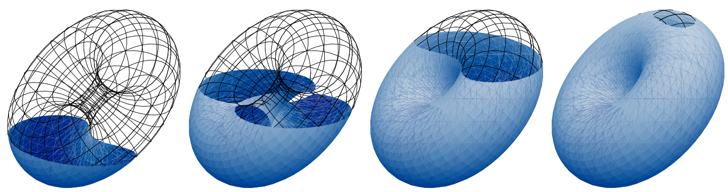

Figure 1: Slicing sequence for a tilted torus. The second slice is formed by two overlapping circles.

In "Torus Triptych" we used $a = \sqrt{2}$ and $b = 1$. This basic torus was rotated to three different positions and then sliced by a horizontal plane at various heights to obtain the three sequences presented. The lower sequence (in blue) has a particularly interesting slice in the second image (Figure 1). Here, the horizontal plane intersects the torus in two overlapping circles of equal radius. This sequence of slices was discussed recently in [5].

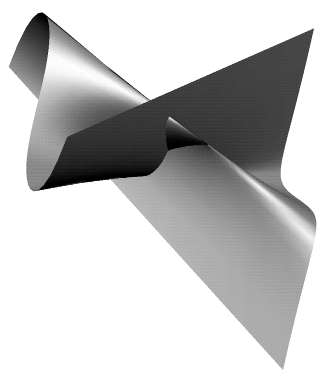

Figure 2: A ruled surface that has self-intersection that forms a triple point.

The surface shown in "Triple-Point Twist" (Figure 2) is one from a series of surfaces described by David Mond and Washington Marar in [4], where they analyze a number of germs of singularities of surfaces. This particular example is a ruled surface given by the equations

$$(x, y, z) = (u, v^3 - cv, uv + v^5 - cv^3) \tag{2}$$

where c is a parameter that can be varied. For values of c greater than 0, the surface has no self-intersection, but for values of c less than 0, a triple point and two pinch points appear. For the image shown in the exhibit, $c = -1$, but an MPEG movie is available showing a series of different values of c as it varies from -1 to 1. For each fixed value of u, allowing v to vary produces a planar curve in the plane $x = u$. The rulings for the surfaces are the straight lines produced when v is held fixed and u is allowed to vary.

While the variable c plays the role of the fourth dimension in the "Triple-Point Twist" and time is the fourth dimension in "Torus Triptych", other surfaces in the exhibit are described originally as objects in four-space and we are presented with three-dimensional projections of them. This is the case in the image titled "Math Horizon", which is a view of a two-dimensional sphere immersed in four-space so that it has exactly one point of self-intersection. To see how this works, first note that surfaces in four-space generally intersect in points rather than in curves as they do in three space. For example, if we label the axes x, y, z, and w, then the xy-plane and the zw-plane are two-dimensional planes in four space, but they intersect in only one point: the origin.

To form the sphere depicted in "Math Horizon", we began by taking the unit disc in the xy-plane and the unit disc in the wz-plane; since they intersect in a single point, these form the essential self-intersection in the surface. The trick now is to attach the boundaries of these two discs so as to form a sphere, and in such a way that no additional self-intersection is produced. The boundaries are two circles, which can be parameterized as $(\cos\theta, \sin\theta, 0, 0)$ and $(0, 0, \cos\theta, \sin\theta)$. For a given θ, these two points, together with the origin, determine a plane in four-space (think of the points as vectors based at the origin that span the plane). For different values of θ, these planes intersect only at the origin, so if, for each θ, we connect the two boundary points by a curve lying in this plane, we will have joined the two disc boundaries to form a sphere with no additional self-intersection, as desired.

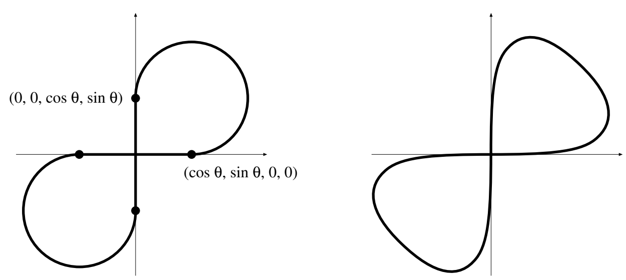

Figure 3: The two points $(\cos\theta, \sin\theta, 0, 0)$ and $(0, 0, \cos\theta, \sin\theta)$ can be joined by circular arcs (left). A smooth figure-eight can replace the piecewise curve (right).

Note that the two points, when considered as vectors at the origin, are perpendicular unit vectors, so they act just like the unit x- and y-axes in the xy-plane. The intersection of the plane spanned by these vectors and one of the discs would be the segment from -1 to 1 along the x-axis, and with the other, the corresponding segment on the y-axis. These two segments form a "cross" at the origin, and one natural way to attach them is by two circular arcs thus forming a figure-eight with an axis of symmetry along the line $y = x$ (Figure 3, left). A piecewise-defined version of the two-sphere in four-space can be produced in this way. On the other hand, we could form a smooth version of the surface if we had a smooth (rather than piecewise-defined) figure-eight (Figure 3, right).

The equation $(\cos t, \sin 2t)$ parameterizes a figure-eight that has the x-axis as an axis of symmetry, though the equation $(\cos t, \frac{1}{2}\sin 2t) = (\cos t, \sin t\cos t) = \cos t\,(1, \sin t)$ is more aesthetically pleasing, as the lobes of the figure-eight are rounder and cross at an angle of 90 degrees. Rotating this curve by 45 degrees about the origin produces a smooth figure-eight with its axis along the line $y = x$ and its crossing tangent to the x and y axes, as desired. Using a standard rotation matrix with angle $\phi = \frac{\pi}{4}$, we obtain

$$\begin{aligned}(x, y) &= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos t \\ \cos t\sin t \end{pmatrix} \\ &= \cos t \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 1 \\ \sin t \end{pmatrix} \\ &= \tfrac{\sqrt{2}}{2}\cos t \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \sin t \end{pmatrix} \\ &= \tfrac{\sqrt{2}}{2}\cos t \begin{pmatrix} 1-\sin t \\ 1+\sin t \end{pmatrix}. \end{aligned} \tag{3}$$

Writing this in vector form, we find

$$(x, y) = \tfrac{\sqrt{2}}{2}\cos t\big[(1-\sin t)(1, 0) + (1+\sin t)(0, 1)\big]. \tag{4}$$

Now, replacing the vectors $(1, 0)$ and $(0, 1)$ by the two vectors from the boundary of the discs in four-space gives a smooth parameterization by t and θ of the two-sphere in four-space with exactly one point of transverse self-intersection:

$$\tfrac{\sqrt{2}}{2}\cos t\big[(1-\sin t)(\cos\theta, \sin\theta, 0, 0) + (1+\sin t)(0, 0, \cos\theta, \sin\theta)\big]. \tag{5}$$

Note that this surface lies within the unit sphere in four-space and touches the unit sphere when $t = 0$, namely along the curve $\frac{\sqrt{2}}{2}(\cos\theta, \sin\theta, \cos\theta, \sin\theta)$, a circle on the four-sphere. The image shown in “Math Horizons” is the stereographic projection of this surface from the point on this circle where $\theta = 0$. Because the surface passes through the point of projection, it’s image appears to extend out to infinity in three-space. Bands of the surface have been removed to help make the structure of the surface and its parameterization more apparent.

Across the gallery from “Math Horizons” is a similar image titled “In- and Outside the Torus”. Unlike the tori in “Torus Triptych”, this is a projection of a torus lying originally in four-space, given parametrically by $(\cos\theta, \sin\theta, \cos\phi, \sin\phi)$. Notice that this represents the cross product of two circles, one in the first two coordinates, and one in the second two. Note also that every point on this torus is at a distance of $\sqrt{2}$ from the origin, so the entire torus lies on the sphere of radius $\sqrt{2}$ in four-space. (In fact, this torus separates the sphere into two congruent solid tori.) Stereographic projection from the point $(0, 0, 0, \sqrt{2})$ in four-space yields an image of the torus in three-space, and since stereographic projection maps circles to circles, the image of the torus given above under this map would be a torus of revolution. In contrast to the parameterization given for the "Torus Triptych", however, this one has the interesting property that it is a conformal mapping of the (θ, ϕ)-plane onto the torus of revolution (Figure 4, left).

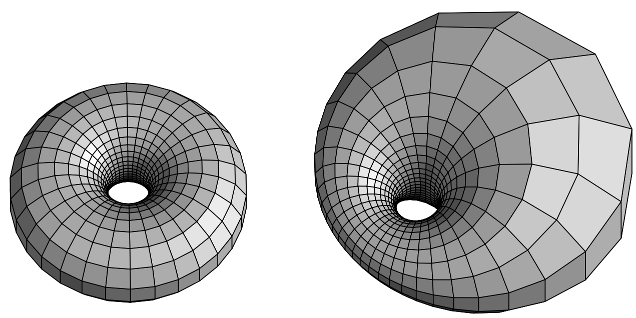

Figure 4: A torus projected from four-space can look like a torus of revolution (left). If part of it is closer to the projection point, part will appear larger, and it will form a cyclide of Dupin (right).

If we rotate the original torus in four-space before projecting it, the image changes: a portion of the torus moves closer to the point of projection, so its image gets larger (just as a shadow gets larger if you move an object closer to the light source), and part of it moves farther away from the point of projection, so its image gets smaller (Figure 4, right). In the projection of the torus, we would see one part of the ring get thicker and the opposite part get thinner. The result is known as a cyclide of Dupin. (The offset surfaces of these projections are also cyclides, and it turns out that every cyclide can be produced in this way.) As the torus rotates further, the torus gets fatter and fatter on one side, and thinner and thinner on the other. After a rotation of 90 degrees, the torus will pass through the point of projection, and so its image will appear to extend to infinity; the images of the two generating circles for the torus will be two infinite, straight lines in three-space. (Two other circles on the torus also map to straight lines in three-space: the $(1, 1)$-curve, described below, that passes through the point of projection, and the analogous $(1, -1)$-curve.) As the torus rotates further, its image again becomes a finite torus, but what was outside the original torus is now inside, and *vice versa*; the torus in three-space has "turned inside out" by passing through infinity.

The image "In- and Outside the Torus" represents the 90 degree rotation, the point of transition when the outside and inside are begin exchanged. Indeed, at this instant, the image of the torus divides all of three-space into two congruent parts, the images of the two solid tori mentioned earlier that form the three-sphere in four-space with this torus as their common boundary. In the picture, the viewer is in one of the two congruent pieces (with the handle of the torus moving horizontally through the center of the picture), and the other piece is "behind" the surface (with the handle moving vertically through the center). A rotation of three-space about a diagonal line from the upper left to the lower right would interchange the two congruent pieces. In the picture, bands on the torus are removed to help make the structure clearer. In this case, the bands are formed by neighborhoods of the $(1,1)$-curves on the torus, which are the images of curves of the form $\theta = \phi + c$ in the (θ, ϕ)-plane.

For those interested in producing similar pictures themselves, we describe the stereographic projection map and rotations in four-space in more detail. Stereographic projection from the point $(0,0,0,d)$ is the map $p_d\colon \mathbf{R}^4 \to \mathbf{R}^3$ given by $p_d(x,y,z,w) = \frac{d}{d-w}(x,y,z)$ for all points where $w \neq d$. In our case, $d = \sqrt{2}$. As with rotations in the xy-plane, rotations in four-space can be represented by matrix multiplication. For example, a rotation in the xw-plane by an angle of ψ is given by the map

$$R_\psi(x,y,z,w) = \begin{pmatrix} \cos\psi & 0 & 0 & -\sin\psi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\psi & 0 & 0 & \cos\psi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}. \tag{6}$$

The composition of these two maps with the parameterization of the torus given above will yield the series of pictures described here (as ψ varies from 0 to 90 degrees and beyond).

The three "Tetraviews" that appear in the gallery are an attempt to better understand graphs of complex functions. The graph of a function of a single complex variable lies in complex two-space, which can be associated in a natural way with real four-space. That is, if $z = x+yi$ and $w = f(z) = u+vi$, then the point $(z, f(z))$ on the graph of f can be though of as (x,y,u,v) in four-space, and so the graph is then a surface in four-space. How can we investigate this surface?

Before answering that question, let's first consider how we can understand an object in three-space, say a cube with corners at $(\pm1, \pm1, \pm1)$. There are three mathematically natural views of this cube in three-space: one looking at it from a point on the positive x-axis, one from the positive y-axis and one from the positive z-axis. From each axis we see a square face of the cube (although a different face in each case). Of course, there are many other views of the cube as well, since there are many other directions from which to view it. One way to think of these directions is by imagining a large sphere enclosing the cube; every point on the sphere represents a different viewpoint, and hence a different view of the cube. The three views we described above are from where the positive coordinate axes intersect the sphere.

These three points form the vertices of a spherical triangle, and we can ask: What does the view look like from different points on this triangle? If we move along an edge of the triangle from one vertex to another, we go from looking at one face of the cube to looking at another. Our intermediate views show one face shrinking down until we see it edge-on and it becomes just a line, while another face that we had been seeing edge-on expands to become a full square. This corresponds to a rotation of the cube about the axis whose vertex is opposite the edge we are traversing; in this way, each edge yields a rotation about one of the axes. Half-way between two vertices on the spherical triangle our view is directly at one of the edges of the cube and both adjacent faces are seen as the same size, though neither looks square at this point (Figure 5, center). If we view the cube from the point at the center of the spherical triangle, we will be looking directly at a corner of the cube, and again we have a symmetric view, but this time including all three faces, still with some distortion (Figure 5, right).

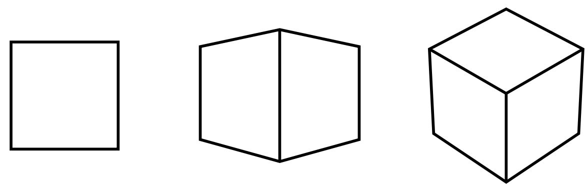

Figure 5: Three views of a cube: looking directly at a face (left), directly at an edge (center), or directly at a corner (right). These correspond to viewpoint located at various spots on a spherical triangle: at a vertex, the center of an edge, or the center of the triangle.

Now suppose the cube is transparent and we place some object inside the cube. Then the three views from the vertices of the spherical triangle give us the three views of the object through the cube's three sides (these are like an architect's three views: the floor-plan from above, the side elevation and the front elevation). As we move along the edges of the spherical triangle, we rotate between these views of the object inside the cube (e.g., moving from the front to the side elevation). Looking from the center of the triangle we can see into all three sides of the cube at once, giving a combination view that is, in a sense, the average of the other three (it corresponds to the architect's perspective drawing of a house).

The tetraviews carry out this same process in four-space. The cube is now a hypercube in four-space (it is transparent so it doesn't appear in the images itself), and the object inside is the graph of a complex function. The different viewpoints lie on a large three-sphere in four-space that contains the hypercube, and since there are four axes, these intersect the sphere at four points. These points form the vertices of a spherical tetrahedron on the three-sphere (thus the name "tetraview"). The four views from the corners of this tetrahedron represent projections of the function graph along each of the coordinate axes. These are shown in the picture (Color Figure 2) at the four corners of the image and are arranged so as to suggest a tetrahedron: two are farther back (lower left and upper right) while two are farther to the front (upper left and lower right). The back corners are the projections into xyu-space and xyv-space, and so represent the graphs of the real and imaginary parts of the function, while the other two corners are projections into xuv- and

yuv-space, which represent the real and imaginary parts of the *inverse* relation for the function. The image at the center of the picture is the view from the center of the spherical tetrahedron, which represents a combination of the other four, the most general view of the surface in four-space.

As with the spherical triangle in three-space, the paths along the edges of the spherical tetrahedron in four-space represent rotations of the surface inside the hypercube. This idea is explored more fully in the article "Understanding Complex Function Graphs" in the prototype issue of the new electronic journal *Communications in Visual Mathematics*, which includes interactive methods of navigating the views from the spherical tetrahedron.

The surfaces shown in the three tetraviews are the complex squaring function $w = z^2$, the complex cubing function $w = z^3$ and the complex exponential function $w = e^z$ (the latter appears as Color Figure 2). To determine the surfaces in terms of the four real coordinates, we use the fact that $z = x+yi$, $w = u+vi$ and $i^2 = -1$. Then for $w = z^2$ we have $w = (x + yi)^2 = x^2 + 2xyi + y^2i^2 = x^2 - y^2 + 2xyi$, so $u = x^2 - y^2$ and $v = 2xy$. This gives the graph parametrically as $(x, y, x^2 - y^2, 2xy)$. The other surfaces are treated similarly.

The images "Z-Squared Necklace" and "Z-Cubed Necklace" also show views of the complex squaring and cubing functions. The sequence begins with the graph of the real part of the function (viewed from above, i.e., from the u-axis, so that what we see is just a disc in the xy-plane) and ending with the graph of the real part of the inverse relation (viewed from the negative x-axis, so we see a doubly or triply covered disc in the uv-plane). The intermediate images show views after rotating the surface in both the the xv- and yu-planes by an angle of θ, for several values of θ between 0 and 90 degrees. As a projection into 2-space, each view shows a three- or four-fold symmetry (Figure 6). The on-line gallery provides movies that give the complete sequence of which the five in each necklace are a part.

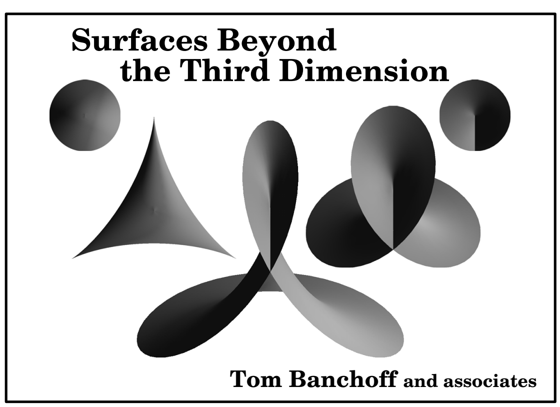

Figure 6: The z-squared necklace as it appeared on the postcard invitation to the gallery show.

One way to see the symmetry is to look at the boundary of the unit disc in the xy-plane. This can be parameterized as $(x, y) = (\cos t, \sin t)$. Since we have already seen that the complex squaring function has the graph $(x, y, x^2 - y^2, 2xy)$, the image of this circle is then $(\cos t, \sin t, \cos^2 t - \sin^2 t, 2\cos t \sin t) = (\cos t, \sin t, \cos 2t, \sin 2t)$. Rotating this through an angle of θ in the xv and yu-planes gives

$$\begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & 0 & \sin\theta \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ -\sin\theta & 0 & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \\ \cos 2t \\ \sin 2t \end{pmatrix}. \tag{7}$$

Multiplying the matrices and then taking the orthogonal projection into the xy-plane gives the curve $(x, y) = (\cos\theta\cos t + \sin\theta\sin 2t, \cos\theta\sin t + \sin\theta\cos 2t)$, which equals

$$(x, y) = \cos\theta(\cos t, \sin t) + \sin\theta(\sin 2t, \cos 2t). \tag{8}$$

Plotting this curve for a fixed θ reveals that it does indeed have the required 3-fold symmetry. It is left as an exercise for the reader to verify that this curve is a hypocycloid formed by a small circle rolling along the inside of a larger circle with radius three times that of the small circle (thus the three-fold symmetry). The point that traces the cycloid may be anywhere along the radius of the small circle (indeed even outside it). In fact, if the radius of the inner circle is normalized to be of unit length, then the point is at a distance of $2\tan\theta$ from the center of the small circle.

5. Generating the Computer Images

Despite the fact that good mathematical software exists, the production of high-quality computer images and movies is still a difficult and time-consuming process. We used a variety of tools to produce the pictures for the gallery show. The "Torus Triptych" was generated using a program called `fnord` developed at Brown University, but not generally available to the public. The "Triple-Point Twist", "In-Outside the Torus" and "Math Horizons" images were created by an ancient piece of custom software for SGI workstations developed by Nick Thompson as an undergraduate at Brown University. Remarkably, it still runs beautifully after more than 10 years without maintenance. The three images in celebration of Dirk Struik's 100th birthday were generated at the graphics laboratory at Brown University. The remaining images were produced using `geomview`, which is distributed as freeware by the Geometry Center at ⟨`http://www.geom.umn.edu/`⟩, though it runs only on unix workstations. The MPEG movies that are part of the interactive gallery were created using `geomview` and its associated `StageTools` modules.

The images were produced first as high-resolution `TIFF` files, but some post-processing was done after creation (e.g., combining the separate images to form the necklace and tetraview sequences) with a variety of image tools on both the unix workstation and on a Macintosh. These tools included the `ImageMagick` library under unix, and `GraphicConverter` on the Macintosh. The final results were printed as Ilfochrome images at 20 by 24 inches and mounted on foam-core. The images in "Torus Triptych" were arranged so that the 20 by 24 prints could be cut in half and joined end-to-end to form 12 by 40 or 10 by 48 panels.

6. Conclusion

It is clear that this electronic gallery faithfully reproduces a great many of the aspects of the actual exhibit, while altering the experience in other ways. In some cases, the electronic version loses information, while in others it provides the opportunity for significant enhancement, particularly in satisfying the viewer's curiosity about the different parts of the mathematics and computer science that made it possible for us to produce these works. We are grateful for the opportunity to present our work in a way that will continue long after the physical exhibit has given way to the work of other artists, and we look forward to further responses from visitors to our virtual gallery.

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