COMPUTER GRAPHICS IN MATHEMATICAL RESEARCH, FROM ICM 1978 TO ICM 2002: A PERSONAL REFLECTION

THOMAS F. BANCHOFF
Brown University, Providence, RI 02912, USA

Six International Congresses ago, a 45-minute invited address in Helsinki introduced the use of computer animation in mathematical research and teaching. Now, twenty-four years later, we can return to the same themes and see how the advent of raster graphics, real-time display, and interactivity on the Internet can enrich our understanding of the geometry of surfaces, both for research and teaching. In the sections below, we follow the outline of the article in the Proceedings of the 1978 ICM, comparing the original text (in small type) to some current developments of the ahuthor and his associates. Our illustrations recreate the wire-frame images from the original article and juxtapose them with modern images using software developed by Davide Cervone. The appendix is a narrative of the circumstances leading up to the presentation in the Helsinki ICM.

Introduction

In the Helsinki ICM, the title of the talk was "Computer Animation and the Geometry of Surfaces in 3- and 4-Space" and that remains the focus of the research and teaching of the author to this day.

Geometers have always used any available media to help them illustrate their work with diagrams, pictures, and models. Modern computer graphics provides a new medium with great potential both for teaching and research. Older methods of representing curves and surfaces by drawings on a blackboard or models in wire or plaster are frequently found to be inadequate in many important geometric problems, specifically those which involve objects undergoing transformations or objects which exist properly in the fourth dimension or higher. A high-speed graphics computer makes it possible to approach and solve such problems by methods that were unavailable only a few years ago.

At the time of the writing of the paper, the "high-speed graphics computer" was a device that would rotate a not very complicated wire-frame model in 3-space in real time. Our system included two improvements that were especially useful for our research—a hardware matrix multiplier designed by Harold Webber that made it possible for us to rotate figures in real time in 4-space, and a routine that could interpolate linearly between wire-frame surfaces with the same parametrization.

The next paragraph of the original article gave an introduction to computer graphics imagery that we take for granted today. There was a short discussion of motion clues versus stereoscopic pairs as the best way to view complicated geometric objects. Then, as now, the impressions created by objects in motion seem to work best for most viewers, especially in large audiences.

We provide a background by rotating a wire-frame model slowly about a vertical axis in space. Subsequent deformations take place within this background. For example the slices of a surface made by planes parallel to a fixed direction appear to the viewer to be planar slices of the rotating figure. Although slicing by a linear function, or more generally by some other function, does require some computational ability, the machine operates quickly enough that it is possible to view a sequence of slices in "real time", as if observing an object through a window as it rotated in the next room.

Modern rendering techniques have largely made the background rotation unnecessary, although we ordinarily provide rotation about a vertical axis as an aid in viewing complicated surfaces. The slicing techniques referred to used a standard "marching squares" algorithm to identifiy intersections of edges with the slicing plane and to connect points on the same rectangle in the domain by segments. Since the parameter domains only included a few hundred rectangular cells, the computer could produce moving slices of a rotating object at a rate fast enough for previewing scenes.

The effect, however, is greater if in addition to the slice, the film displays as well the part of the surface lying below or above the slice—the technique of "water-level slicing". Since this technique requires more time per picture and since it is especially well suited to representations using color, this technique is used primarily in the animation mode, where one picture is made at a time, and then the finished film is projected at 24 frames per second. For color, filters are used and each frame is exposed several times for the different portions of the picture.

It is hard to imagine computer workstations without color these days, but in the seventies they were rare. Multiply exposing each frame of a film to achieve a short sequence in color was intensely laborious, but the results were a striking preview of what we have today.

In addition to projection, rotation, and slicing it is possible to use linear interpolation between figures with the same parametrization. Again this is fast enough that the technique can be employed for real time manipulation of the figures for videotapes or for on-line research.

Video technology was extremely primitive at this time. There were competing standards and sizes of videotape, and it was nearly impossible to show a tape unless you brought the equipment with you, and even then it was impossible to project them for a large audience. Although standards took place very quickly in the United States, it is still complicated to show a videotape in other countries.

The Hypercube: Projections and Slicing

This film treats the convex hull of the sixteen points $(\pm1, \pm1, \pm1, \pm1)$ in 4-space, first by orthogonal projection then by central projection from 4-space to 3-space. In each case we rotate in the coordinate planes xy, yu, xw, yw, and zw, ending at the original position. We then slice each figure by hyperplanes perpendicular to the vectors $(1,0,0,0)$ then $(1,1,0,0)$ then $(1,1,1,0)$ and finally $(1,1,1,1)$. For a more thorough description of this film, see Banchoff.[4]

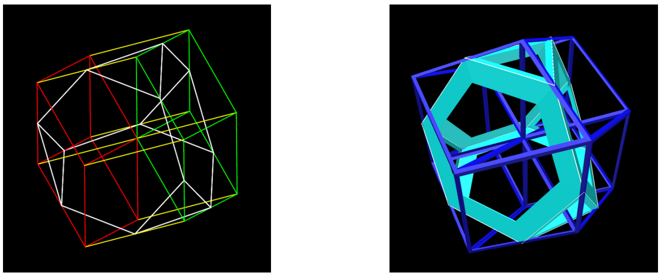

Figure 1. A frame from "The hypercube: projections and slices", and a modern version from the artwork "Iced cubes".

In these days, programming images of hypercubes is a beginning exercise in introductory courses in computer graphics. Instead of demanding a workstation, it is possible to realize scenes on a laptop computer. Even then the topic had been treated by several researchers, most notably A. Michael Noll at Bell Laboratories.[24] Our main contribution was a scripted tour of the 4-dimensional cube with three movements, orthographic projections, central projections, and slicing by planes and hyperplanes. After a quarter of a century, this film, now available in video, is still in demand, especially in schools and colleges. For modern interactive versions of the object, see for example the two cube sequences in the interactive art exhibit site "Para Álem da Terceira Dimenšao".[16,17] The reference[4] is a short description in an article where the illustrations came from Polaroid pictures taken directly off the computer screen!

Complex Function Graphs

This film treats graphs of complex functions $w = f(z)$ considered as parametric surfaces (x,y,u,v) in 4-space, where $z = x + iy$ and $w = u + iv$. In each case orthographic projection into (x,y,u) is used to get the graph of the real part of w, then rotation in the uv plane gives (x,y,v), the graph of the imaginary part of w. Rotating the original graph in the xv plane leads to (y,u,v), the graph of the imaginary part of the inverse function of f. Finally, projection to (x,u,v) gives the graph of the real part of the inverse function (Figure 2).

A particularly interesting example is the exponential function $w = e^z$ with the inverse relation $z = \log(w)$. The graph is given by $(x, y, e^x \cos(y), e^x \sin(y))$ in 4-space. Projection to (x,y,u) gives the real part of the exponential (Figure 2, left). The projection (y,u,v) gives a right helicoid which represents the imaginary part of the Riemann surface for the logarithm (Figure 2, center). The projection (x,u,v) gives a surface of revolution of a real exponential function as the real part of the logarithm.[3]

Color certainly is a considerable help in analyzing these objects. Coloring the axes and displaying a coordinate frame during rotations is a very effective way of keeping track of positions in real four-dimensional space. The authors have devised other methods for displaying complex function graphs, both of which have played central roles in expositions of art and mathematics, most recently in Lisbon.[16]

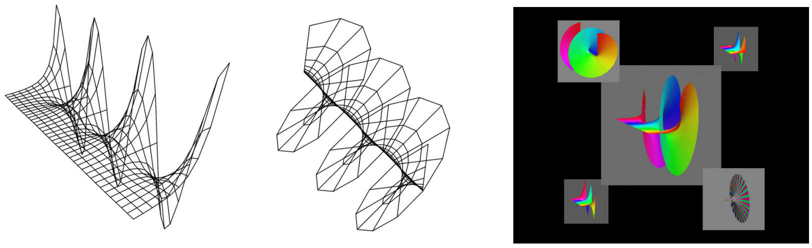

Figure 2. Graphs of the real part of the exponential function and the imaginary part of the logarithm, together with a "tetraview" for the exponential function.

A method with promise is the "tetraview", a way of displaying the real and imaginary parts of a function and its inverse relation in such a way that we can see the deformation from any position to any other by selecting edges in the one-skeleton of a tetrahedron whose vertices represent the four main projections from 4-space. We can also consider the intermediate position at the barycenter of the tetrahedron, the average of the four extreme views.

The Gauss Map, A Dynamical Approach

In the Helsinki ICM, the film "The Gauss Map: A Dynamical Approach" was a preliminary version that led to the 1982 monograph "Cusps of Gauss Mappings"[7] with Clint McCrory and Terence Gaffney. The illustrations in that volume were wire-frame images similar to those in the Proceedings article. Subsequently that monograph has appeared in an online updated version[8] with highly rendered images produced by Dan Dreibelbis, using software developed by Davide Cervone.[20]

Both the original article and the monograph presented of one-parameter families of surfaces involving the appearance and disappearance of cusps on the Gauss mapping. One of them, a perturbation of the "monkey saddle", exhibits a disc on the surface where the Gaussian curvature is positive such that the spherical image map has three cusps on the boundary curve. This provided an example that clarified a construction of Réné Thom.[6]

We consider the monkey saddle, with an isolated point of zero Gaussian curvature, and perturb to get the graph of $(x, y, x^3 - 3xy^2 + k(x^2 + y^2))$. For $k = 0$, this surface has a Gauss mapping with a ramification point of order 2, and for k not zero, the image of the parabolic curve will have three cusps (Figure 3).

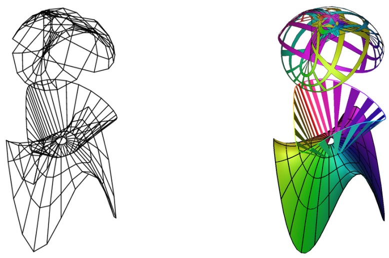

Figure 3. The Gauss map of a perturbed monkey saddle.

We show the spherical image of a circle $x^2 + y^2 = r^2$ as r changes. We show the linear interpolation between the surface and its Gauss spherical image so that the singularities of the Gauss map are expressed as limits of singularities of homothetic images of parallel surfaces of the original surface. We then show the spherical image of a test curve centered on the curve $r =$ constant and indicate the behavior of the asymptotic vectors in a neighborhood of a cusp of the Gauss mapping.

Included in the previous paragraph is one of the most striking applications of linear interpolation between surfaces with the same parametrization. If we consider the family $uX + (1 - u)N$ interpolating between a surface X and its Gauss mapping, then this can be viewed as a scaled version $u(X + (1-u)/uN)$ of the parallel surface at distance $r = 1/u - 1$. Thus the Gauss mapping can be thought of as "the parallel surface at infinity".

Various characterizations of the singularities of the Gauss map in terms of lines of curvature, ridges, and double tangencies are included in "Cusps of Gauss Mapping". Using the linear interpolation between a surface and its spherical image gives an interpretation of a cusp of the Gauss mapping as a limit of swallowtail points on the scaled parallel surfaces.

The Veronese Surface

The Veronese surface is an embedding of the real projective plane which starts with the hemisphere $x^2 + y^2 + z^2 = 1$, $z \leq 0$ and maps each point (x, y, z) to 6-space. The projection of this surface $(x^2, y^2, z^2, \sqrt{2}xy, \sqrt{2}yz, \sqrt{2}zx)$ from 6-space into 4-dimensional space given by $(\sqrt{2}xz, \sqrt{2}yz, (1/\sqrt{2})(x^2 - y^2), \sqrt{2}xy)$ is again an embedding and we examine a family of projections of this surface into 3-dimensional subspaces (all of which must have local singularities). This will appear in a paper on normal Euler classes by the author.

The paper referred to was published, in expanded form, as a collaboration with Ockle Johnson[11] twenty years after the Helsinki Congress! In the meantime, several other papers of the author have used the fact that the normal Euler class of a surface embedded in four-space can be obtained as an indexed sum of singularities of any generic orthogonal projection into a hyperplane. Since the normal Euler class of an embedded real projective plane is non-zero, there must be singular points for almost any such projection.

The projection into the last three coordinates gives a cross-cap with two pinch points (Whitney umbrella points). The linear interpolation of the left hemisphere into the cross-cap is a regular homotopy right up to the last instant when opposite points on the equator are identified, forming a segment of double points.

Deforming a hemisphere into a cross-cap is a another remarkable use of linear interpolation between surfaces with the same parametrization. It is a challenge, however, to position the surfaces so that the intermediate stages are all embedded with two-fold symmetry. One way is to interpolate between the cross-cap

$$\frac{1}{\sqrt{2}}(\sin u \sin v, \frac{1}{2}(\cos 2u)(1+\cos 2v), \frac{1}{2}(\sin 2u)(1+\cos 2v))$$

and the hemisphere given by $(\sin v, -\sin u \cos v, \cos u \cos v)$ where u goes form 0 to π and v goes from -$\pi/2$ to $\pi/2$.

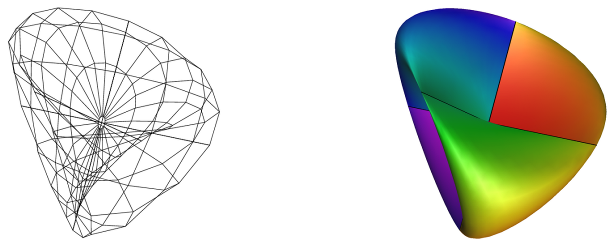

Figure 4. The projection of the Veronese surface as Steiner's Roman surface.

Rotating in the plane of the first and third coordinates gives a deformation from the cross-cap to Steiner's Roman surface $(\sqrt{2}xz, \sqrt{2}yz, \sqrt{2}xy)$ (Figure 4) with tetrahedral symmetry. This projection has six pinch points that are the end-points of three double point segments intersecting in a triple point. These examples are described in the classical book, "Geometry and the Imagination" by Hilbert and Cohn-Vossen.[23]

The embedding in 4-space is tight (i.e. almost every height function when restricted to the surface has exactly one maximum and one minimum) and this property is shared by the images in 3-dimensional subspaces. These examples lead to the conjecture that any stable tight mapping of the real projective plane into 3-space must have either two pinch points or six pinch points.

This conjecture was established in the Ph.D. thesis of Leslie Coglan, under the author's direction.[21,22]

The Flat Torus in the 3-Sphere

The flat torus is an embedding as a product of two circles in 4-space considered as the product of two planes, i.e. $(\cos u, \sin u, \cos v, \sin v)$. This torus is a surface on the 3-sphere of radius $\sqrt{2}$. We may project stereographically from $(0, 0, 0, \sqrt{2})$ to obtain a torus of revolution in 3-space.

Rotating the flat torus in the plane of the first and fourth coordinates produces a one-parameter family

$$(\cos a \cos u + \sin a \sin v, \sin u, \cos v, -\sin a \cos u + \cos a \sin v).$$

These rotated images of the flat torus project to a family of cyclides of Dupin, all conformally equivalent to the original torus. In particular when $a = \pi/2$ and the point $(0, 0, 0, \sqrt{2})$ is on the rotated torus, the projected surface is a non-compact cyclide which separates all of 3-space into two congruent parts.

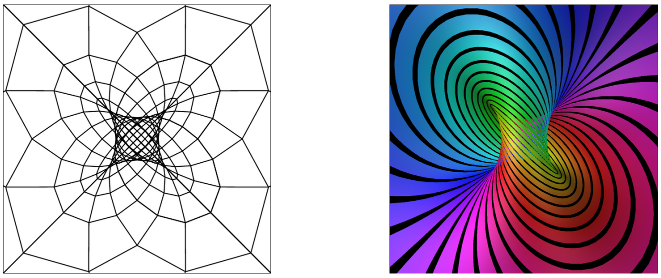

Figure 5. Projection of a torus in 4-space from a point on the surface, so that the projection extends to infinity, separating space into two congruent regions.

This sequence was the first film we ever made, in 1968–9. Stills from the sequence have been recreated several times, in the paper with David Laidlaw, Fred Bisshopp, and Hüseyin Koçak on Hamiltonian Dynamical Systems,[18] in the related film by these authors and David Margolis, and on the cover of "Beyond the Third Dimension".[13] The flat torus was the first example to be developed by Davide Cervone and the author at the Geometry Center of the University of Minnesota in 1994, and it is one of the featured items in both our Providence Art Club and Lisbon exhibits.[15,16]

The cyclides of Dupin and spheres are the only closed surfaces in 3-space that have the spherical two-piece property, i.e. any sphere separates them into at most two pieces. Their inverse stereographic projections are the only surfaces on the 3-sphere that are tight, i.e. every hyperplane separates them into at most two pieces.[1]

Tight and taut surfaces and their generalization have undergone extensive development over the past twenty-five years. A full account of the subject can be found in the volume edited by Shiing-Shen Chern and Thomas Cecil, including a posthumous article by Nicolaas Kuiper and an extensive survey by Wolfgang Kühnel and the author.[12]

Appendix

How did it come about that an invited 45-minute address on computer graphics in mathematical research occurred at the International Congress of Mathematicians in Helsinki in 1978? What did we know at that time and what did we expect? What has actually happened over the past 24 years with respect to the topics featured in that first computer-illustrated invited talk at an ICM? This report at the ICMS will address the changes that have taken place in the author's collaborations with colleagues and students, particularly concerning advances in software for research and teaching in the areas of geometry and topology.

In the early 1970s, the only medium for presenting computer animation to an audience was film. In 1967, when I arrived at Brown University, I was fortunate to meet an applied mathematician, Charles Strauss, just finishing his Ph.D. in three-dimensional computer graphics, the first doctoral thesis supervised by the world expert on the subject, Andries van Dam. We began using the techniques from his thesis to analyze surfaces in four-dimensional space, projected orthographically or stereographically into three-space, and within one year, we had produced our first film, "The Hypertorus" (described above). It was a laborious process, waiting the better part of a minute for each image of a wire-frame model with 400 vertices, then taking two pictures and instructing the computer to produce the next image. It took all night in a darkened room before a storage tube to generate a few seconds worth of film. What we saw convinced us that not only was it worth the effort. We would never be satisfied again with still images.

It was nearly five years before we succeeded in producing our next films, preliminary versions of "The Hypercube: Projections and Slicing" and "Complex Function Graphs". I showed these in a two-part series at Berkeley during my first sabbatical, 1973–4. The first talk was titled "*Neue Polyhedralische Methoden in der Differentialgeometrie*" and the second was "*Disquisitiones Generales super Superficies Polyhedrales*". My Ph.D. advisor, Professor S.-S. Chern, showed interest in the project.

Two years later I gave a presentation at a symposium in honor of Prof. Chern. Afterwards he said to me that he thought it was time to present this to the world. He would put in a word, he said, with the selection committee for the ICM coming up in Helsinki. In late summer, 1977, I received a call from Prof. Armand Borel at the Institute for Advanced Study in Princeton. Could I tell him something about these films that had been mentioned? I said I would be coming to New Jersey in two weeks for a talk and I could show some films to him then. He said he would have to know about them before that, and perhaps I could describe them on the phone. "I'll be there tomorrow," I said. Prof. Borel put up an announcement, and over forty people arrived for a presentation in the IAS theater. I narrated films on the hypercube, the Gauss mapping, and an early version of the Veronese surface, and presented two films on complex functions with their own soundtracks. The audience was appreciative.

Afterwards in his office, Prof. Borel said he thought that it would be good to have a showing of these films at the ICM. He indicated that there could be a side event some evening, or there could be an invited talk in Section XIX, on History and Pedagogy. "You would probably prefer the latter?" he asked. I agreed, and that was the invitation.

Charles Strauss and I worked hard that next year, producing the version of "The Hypercube: Projections and Slicing" that is still used to this day. Since there were no color monitors available at that time, we achieved color by using color filters and a quadruple exposure for each frame, a semi-automated process that was still quite labor-intensive. The other films were all black-and-white, using vector graphics images with no filled surfaces. We worked especially on the Veronese surface, a series of images of the real projective plane which some exceptional singular projections.

At the ICM in Helsinki in 1978, the invited presentation was early in the meeting and over four hundred mathematicians attended. A few days later, Prof. Michael Atiyah asked if I would give another showing since many people had heard about the films only after the address. Another full audience watched a second showing, in which I included the original "Hypertorus" film. The article for the Proceedings used screen photographs of stills from the films as illustrations. Most of the article is reproduced above, along with annotations and new renditions of many of the images.

The purpose of the talk in Helsinki was to show how this new medium could offer new insights and new possibilities for teaching and research. In this article, we make particular mention of advances that have taken place using computer graphics in various aspects of mathematics, especially when visualization contributes new insights into geometric phenomena, new conjectures, and new methods of proof.

References

1. Banchoff, Thomas F., The spherical two-piece property and tight surfaces in spheres. *J. Differential Geometry* **4** (1970), 193–205.
2. —–, with Charles Strauss, Computer-animated four dimensional geometry, *Proc. Amer. Assoc. Advancement of Science*, Washington, 1978.
3. —–, with Charles Strauss, Real-time computer graphics techniques in geometry, The Influence of Computing on Mathematical Research and Education, *Proc. Sympos. Appl. Math.*, vol. 20, Amer. Math. Soc., Providence, R. I., 1974, pp. 105–111.
4. —–, with Charles Strauss, Real-Time Computer Graphics Analysis of Figures in Four-Space, *American Association of the Advancement of Science Selected Symposium* **24** (1978), Westview Press, Colo., 159–168.
5. —–, Computer animation and the geometry of surfaces in 3- and 4-space, Proceedings of the International Congress of Mathematicians (Helsinki, 1978), Acad. Sci. Fennica, Helsinki, 1980, 1005-1013.
6. —–, with Réné Thom, Sur les points paraboliques des surfaces: erratum et complements, *C. R. Acad. Sc. Paris* **291** (27 Octobre 1980), 503–505.
7. —–, with Terence Gaffney and Clint McCrory, Cusps of Gauss mappings, *Research Notes in Mathematics* **55**. Pitman (Advanced Publishing Program), Boston, Mass., London, 1982.
8. —–, with Terence Gaffney, Clint McCrory and Dan Dreibelbis, Cusps of Gauss mappings, the Electronic Version, European Mathematical Information Service, ⟨`http://www.emi.de/monographs/CGM`⟩, 2000.
9. —–, Normal Curvatures and Euler Classes for Polyhedral Surfaces in 4-Space, *Proc. A.M.S.*, 92 (1984), 593–596.
10. —–, Differential Geometry and Computer Graphics, *Perspectives in Mathematics*, Anniversary of Oberwolfach (1984) Birkhäuser-Verlag,

Basel, 43–60.

11. ——, with Ockle Johnson, The normal Euler class and singularities of projections for polyhedral surfaces in 4-space, *Topology* **37** (1998) no. 2, 419–439.
12. ——, with Wolfgang Kühnel, Tight submanifolds, smooth and polyhedral, *Tight and taut submanifolds* (Berkeley, CA, 1994), 51–118, *Math. Sci. Res. Inst. Publ.*, **32**, Cambridge Univ. Press, Cambridge, 1997.
13. ——, *Beyond the third dimension: Geometry, computer graphics, and higher dimensions.* Amended and enhanced version of the 1990 original. Scientific American Library Paperback, 33. *Scientific American Library*, New York, 1996.
14. ——, with Davide P. Cervone, Illustrating "Beyond the Third Dimension", *The visual mind*, 85–92, *Leonardo* Book Ser., MIT Press, Cambridge, MA, 1993.
15. ——, with Davide P. Cervone, *Surfaces Beyond the Third Dimension*, virtual art gallery, 1996; ⟨`http://www.math.brown.edu/~banchoff/art/PAC-9603/`⟩.
16. ——, with Davide P. Cervone, *Para Álem da Terceira Dimensão*, physical and virtual art exhibit, Lisbon, 2000; ⟨`http://alem3d.obidos.org/`⟩.
17. ——, with Davide P. Cervone, A virtual reconstruction of a virtual exhibit, in *Multimedia Tools for Communication Mathematics*, Springer-Verlag, Berlin, Heidelberg, 2002, 29–38.
18. ——, with H. Koçak, F. Bisshopp and D. Laidlaw, Topology and Mechanics with Computer Graphics: Linear Hamiltonian Systems in Four Dimensions, *Advances in Applied Mathematics* (1986), 282–308.
19. ——, *Beyond the Third Dimension*, (1990) New York: W. H. Freeman & Co., Scientific American Library, 1–210.
20. Cervone, Davide P., `StageTools` Geometry Package, ⟨`http://www.math.union.edu/locate/StageTools`⟩.
21. Coghlan, Leslie, Tight stable surfaces, I, *Proc. Roy. Soc. Edinburgh Sect. A* **107** (1987) no. 3–4, 213–232.
22. Coghlan, Leslie, Tight stable surfaces, II, *Proc. Roy. Soc. Edinburgh Sect. A* **111** (1989) no. 3–4, 213–229.
23. Hilbert, D. and Cohn-Vossen, S., *Geometry and the imagination*, Chelsea, New York, 1952.
24. Noll, A. Michael, A Computer Technique for Displaying n-Dimensional Hyperobjects, *Communications of the ACM*, **10** (1967) no. 8, 469–473; Reprinted in *Hypergraphics*, D. Brisson, ed., American Association for the Advancement of Science Selected Symposium 24 (1978) Westview Press, Colo., 159–168.